Historical Notes on Navier-Stokes Equations, I: Basic Settings and Local Well-posedness Theory

Ben Hou

November 2021

Abstract

In this artical, we get a quick review of basic settings of Navier-Stokes equations and its fundamental local well-posedness theory.

Contents

1 Introduction

We focus on the *incompressible Navier-Stokes equations*, which is of the form

$$\begin{aligned} \partial_t u + (u \cdot \nabla) u &= -\nabla p + \nu \Delta u \\ \nabla \cdot u &= 0, \end{aligned} \tag{NSE}$$

where $u : I \times \mathbf{R}^d \to \mathbf{R}^d$ is an unknown vector field, called the *velocity field*, $p : I \times \mathbf{R}^d \to \mathbf{R}$ is an unknown scalar field, called the *pressure field*, and $\nu > 0$ is a fixed constant, often called *viscosity*. Here, I is often of the form $[0, T]$ or $[0, T)$ for some $T > 0$, regarded as the domain of *time* variable t, and *spatial* variables xs are defined on $\mathbf{R}^d$ as usual. Tautologically, one can always ask whether it can be established on the inverse time direction, but the diffusion term tells us that it may not make sense.

The case for $\nu = 0$ in Eq.(NSE) is called the *incompressible Euler equations*.

1.1 The Cauchy problem of NSE

Regarding the system as an evolutionary system for the velocity field u with respect to time, the natural problem is to determine the flow after obtaining the initial value

$$u(x,0) = u_0(x) \tag{IVP}$$

for any $x \in \mathbf{R}^d$. For compatibility, we need that $\nabla u_0 = 0$.

According to the complexity of the system, we are forced to restrict ourselves with much simpler boundary value conditions, for instance, periodic boundary value condition, $u(t, x+n) = u(t,x)$ for any $n \in \mathbf{Z}^d$, or spatially decaying condition, $\lim_{x\to\infty} u(t,x) = 0$. We shall see that these two systems share a series of most significant properties.

The local theory of a PDE is established on answering the following problems revealing different levels of behaviours of the solution:

1. (Local existence) Provided with an initial value $u(x,0) = u_0(x)$, does there exist an solution (u,p) for some $T > 0$? What is the extreme existence case for such T?

2. (Uniqueness) Provided with the same initial value, would it be possible to find two different systems $(u,p), (u',p')$? Or may we find an equivalence relation between such two systems so that the uniqueness still holds up to an equivalence class?

3. (Continuous dependence on initial data) What if we perturb the initial data u_0 by a tiny change? How much does it violate the behaviour of such a system?

Note that the whole problems are established on the certain "*regularity*", that is to say, the level of smoothness given on the initial value. A system is called to be "*locally well-posed*" if all the answers to the above problems are successfully established, and such a theory is called a "*strong*" theory.

Nevertheless, the moral reminds us that lowering the regularity is **not** equivalent to getting rid of the strong properties. Modern techniques on PDEs tend to discover a low-regularity solution after loosing the classical restrictions, such as differentiability, of solutions to construct a "*weak*" theory in turn, and then find the "*weak-strong correspondence*" to pinpoint the solution into high-regularity spaces.

1.2 Decomposition of the system

Three types of PDEs are mixed in the NSE:

1. *Transport equations* as $\partial_t u + (u \cdot \nabla u) = 0$.

2. *Heat equations* as $\partial_t u = \nu \Delta u$.

3. *Leray systems* as $v = F - \nabla p, \nabla \cdot v = 0$.

Let us start from the most unfamiliar one.

1.2.1 Leray systems

Let $F : \mathbf{R}^d \to \mathbf{R}^d$ be a given vector field. Consider the system

$$\begin{aligned} v &= F - \nabla p \\ \nabla \cdot v &= 0, \end{aligned} \tag{Leray-S}$$

where $v : \mathbf{R}^d \to \mathbf{R}^d$ is a vector field, and $p : \mathbf{R}^d \to \mathbf{R}$ is a scalar field. Heuristically, the system can be solved by taking gradient and the inverse of Laplacian as

$$\begin{aligned} p &= \Delta^{-1}(\nabla \cdot F) \\ v &= F - \nabla\Delta^{-1}(\nabla \cdot F). \end{aligned} \tag{1}$$

Rigorous deduction relies on the smoothness and the periodic boundary condition as follows.

Proposition 1. *Suppose that $F : \mathbf{R}^d/\mathbf{Z}^d \to \mathbf{R}^d$ is smooth. Then, Eq.(1) is indeed a periodic solution to Eq.(Leray-S) based on the definition*

$$\Delta^{-1} f(x) := \sum_{k \in \mathbf{Z}^d \setminus \{0\}} -\frac{1}{4\pi|k|^2} \widehat{f}(k) e^{2\pi i k \cdot x} \tag{2}$$

for any $f : \mathbf{R}^d/\mathbf{Z}^d \to \mathbf{R}$.

Proof. We first concentrate on the uniqueness. Linearity gives that different solutions to Eq.(Leray-S) are equivalent up to the solution set to the system

$$\begin{aligned} v &= -\nabla p \\ \nabla \cdot v &= 0. \end{aligned}$$

The solution is obviously given by a harmonic function as p and its gradient as v. However, Liouville's theorem gives that a periodic harmonic function is identically constant, so the mere possibility for differences on solutions to Eq.(Leray-S) happens on the constant variation on p.

Then turn into existence. Expand F, v, p as Fourier series and Eq.(Leray-S) turns out to be as

$$\begin{aligned} \widehat{v}(k) &= \widehat{F}(k) - 2\pi i k \widehat{p}(k) \\ 2\pi i k \cdot \widehat{v}(k) &= 0, \end{aligned}$$

for any $k \in \mathbb{Z}^d$.

For $k = 0$, $\widehat{v}(0) = \widehat{F}(0)$ and $\widehat{p}(0)$ is arbitrary, which coincides exactly with the assertion before on the uniqueness. For $k \neq 0$, simple computation shows that

$$\widehat{p}(k) = \frac{k}{2\pi i |k|^2} \widehat{F}(k), \quad \widehat{v}(k) = \widehat{F}(k) - k \left(\frac{k}{|k|^2} \cdot \widehat{F}(k) \right).$$

Plancherel's principle hence gives that v, p are well-defined and smooth. $\square$

Remark. It is obvious to verify that for any smooth $f : \mathbf{R}^d/\mathbf{Z}^d \to \mathbf{R}$, $\Delta\Delta^{-1}f = f$.

Remark. A more general case is to assume that F, v are periodic but not necessarily for p. Since the gradient of a harmonic function is still harmonic, the solution to Eq.(Leray-S) is now up to an affine-linear function, which is harmonic and its gradient is indeed a constant.

Remark. The restriction can be loosen as $F \in H^s(\mathbf{R}^d/\mathbf{Z}^d; \mathbf{R}^d)$ via the Solobev norm

$$\|f\|_{H^s(\mathbf{R}^d/\mathbf{Z}^d;\mathbf{R}^m)} := \left(\sum_{k\in\mathbf{Z}^d} \langle k\rangle^{2s} |\widehat{f}(k)|^2\right)^{\frac{1}{2}}.$$

Norms of solutions are bounded as

$$\|v\|_{H^s(\mathbf{R}^d/\mathbf{Z}^d;\mathbf{R}^d)} \lesssim \|F\|_{H^s(\mathbf{R}^d/\mathbf{Z}^d;\mathbf{R}^d)}, \quad \|p\|_{H^s(\mathbf{R}^d/\mathbf{Z}^d;\mathbf{R})} \lesssim \|F\|_{H^s(\mathbf{R}^d/\mathbf{Z}^d;\mathbf{R}^d)},$$

where $A \lesssim B$ means that $A \le CB$ for some constant C independent of A and B. Otherwise, the dependence is given in the subscript.

Remark. It can also be established on the non-periodic but decaying case via Fourier transformation

$$\widehat{f}(\xi) = \int_{\mathbf{R}^d} f(x)e^{-2\pi i\xi\cdot x}dx.$$

and the Sobolev norm

$$\|f\|_{H^s(\mathbf{R}^d;\mathbf{R}^m)} := \left(\int_{\mathbf{R}^d} \langle \xi\rangle^{2s} |\widehat{f}(\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

The indication is, as convention, to consider the Schwartz class first. Without much effort, we can also verify that in such case, it still holds that

$$\|v\|_{H^s(\mathbf{R}^d;\mathbf{R}^d)} \lesssim \|F\|_{H^s(\mathbf{R}^d;\mathbf{R}^d)}, \quad \|p\|_{H^s(\mathbf{R}^d;\mathbf{R})} \lesssim \|F\|_{H^s(\mathbf{R}^d;\mathbf{R}^d)},$$

1.2.2 Leray projection

Definition 1.1. The operator $\mathbb{P} : H^s(\mathbf{R}^d; \mathbf{R}^d) \to H^s(\mathbf{R}^d; \mathbf{R}^d)$ such that

$$\mathbb{P}(F) = F - \nabla\Delta^{-1}(\nabla \cdot F)$$

is called the *Leray projection.*

The following decomposition gives the intuition of Leray projection.

Theorem 2 (Hodge's decomposition)**.** *Define the following three subspaces of the Hilbert space $L^2(\mathbf{T}^d; \mathbf{R}^d)$.*

1. *The space $d\mathcal{E}(\Omega^0)$ contains all elements of the form ∇f for some $f \in H^1(\mathbf{T}^d; \mathbf{R}^d)$ in the sense of distributions.*

2. *The space $\mathcal{H}^1$ contains all harmonic functions in the sense of distributions.*
3. *The space $d^*\mathcal{E}(\Omega^2)$ contains such elements $u = (u_1, \ldots, u_d)$ of the form*

$$u_i = \partial_j \omega_{ij}$$

in the sense of Einstein's summation convention for some tensor $(\omega_{i,j})_{1\le i,j\le d} \in H^1(\mathbf{T}^d; \mathbf{R}^{d^2})$ obeying the anti-symmetry as $\omega_{ij} = -\omega_{ji}$.

Then the followings hold.

1. *They are closed subspaces and we have the orthogonal decomposition as*

$$L^2(\mathbf{T}^d; \mathbf{R}^d) = d\mathcal{E}(\Omega^0) \perp \mathcal{H}^1 \perp d^*\mathcal{E}(\Omega^2).$$

2. *The Leray projection is exactly the orthogonal projection onto $\mathcal{H}^1 \perp d^*\mathcal{E}(\Omega^2)$. In other words, it preserves the divergence-free components.*
3. *The Leray projection is non-expansive on $H^s(\mathbf{T}^d; \mathbf{R}^d)$ for all $s \ge 0$.*

2 Main results

Let us get back to the Navier-Stokes equations. Consider the system Eq.(NSE) on the space-time region $[0, T] \times \mathbf{R}^d$. We first focus on the case that $u : [0, T] \times \mathbf{R}^d \to \mathbf{R}^d$ is spatially periodic as $u(t, x + n) = u(t, x)$ for any $n \in \mathbf{Z}^d$. Most of the results listed here come from the splendid article by Jean Leray [Ler34].

2.1 Local uniqueness theory

Suppose that p is smooth but without any periodic hypotheses, but ∇p is periodic, so $p(t, x+n) - p(t, x) = a_n(t)$ for all $x \in \mathbf{R}^d$. Induction shows that the map $n \mapsto a_n(t)$ is a homomorphism, i.e., $a_n(t) = n \cdot a(t)$ for some $a : [0, T] \to \mathbf{R}^d$. In particular, it gives the periodicity of $p(t, x) - x \cdot a(t)$. Separating the mean, we obtain the model that

$$p(t, x) = x \cdot a(t) + p_1(t, x) + r(t),$$

where $p_1(t, x) : [0, T] \times \mathbf{T}^d \to \mathbf{R}$ is of zero mean and $r : [0, T] \to \mathbf{R}$ gives the original spatial average.

Let us start simplification. Note that $r(t)$ can be directly omitted, and then we use the "generalised Galilean transformation" as

$$v(t) = \int_0^t a(s)ds, \quad X(t) = \int_0^t v(s)ds,$$
$$u_2(t, x) = u(t - X(t)) + v(t), \quad p_2(t, x) = p_1(t, x - X(t)),$$

which also solves Eq.(NSE).

The above discussion shows the reason for lose of uniqueness: the generalised Galilean transformation of x adding the spatially affine function on p. Therefore, after we require that p is periodic and of mean zero, heuristically, such uniqueness is hence preserved up to such an equivalence relation. The following proposition exactly reveal such a fact.

Theorem 3 (Uniqueness up to normalised pressure)**.** *Let* $(u_1, p_1), (u_2, p_2)$ *be two smooth periodic solutions to Eq.*(NSE) *on* $[0, T] \times \mathbf{R}^d$ *with normalised pressure to the initial data* $u_1(0) = u_2(0)$*. Then* $(u_1, p_1) = (u_2, p_2)$*.*

Proof. Conventional trick by subtraction and considering the energy $E(t) = \int_{\mathbb{T}^d} |(u_1 - u_2)(t, x)|^2 dx$. Computation shows that

$$\partial_t E(t) \leq 2E(t) \|\nabla u_2\|_{L_t^\infty L_x^\infty}.$$

Grönwall's inequality directly gives that $u_1 = u_2$ and $\nabla p_1 = \nabla p_2$. The mean zero assumption provides the Fourier inversion formula with the fact that $p_1 = p_2$. □

2.2 Local existence theory

Suppose that (u, p) is a solution to Eq.(NSE) with normalised pressure. We first reduce the pressure field p.

2.2.1 Leray modification

Note that $((u \cdot \nabla)u)_i = u_j \partial_j u_i$ with Einstein's summation convention on the same variable j, i.e., $\partial_j(u_j u_i) = u_j \partial_j u_i$. Define the tensor $u \otimes u$ as

$$(u \otimes u)_{ji} := u_j u_i,$$

so $(u \cdot \nabla)u = \nabla \cdot (u \otimes u)$, where $(\nabla \cdot (u \otimes u))_i = \partial_j (u \otimes u)_{ji}$. The system hence turns out to be as

$$\partial_t u + \nabla \cdot (u \otimes u) = \nu \nabla u - \nabla p.$$

Then apply the Leray projection on it. Note that $\partial_t u$ and ∇u survived since they are both divergence-free. The annihilation makes the NSE be as

$$\partial_t u + \mathbb{P}(\nabla \cdot (u \otimes u)) = \nu \nabla u. \tag{Leray-NSE}$$

Note that the divergence-free restriction is also included in such an equation.

To be precise, we then show that the pressure field can be recovered from solutions to Eq.(Leray-NSE). Suppose that one has a solution $u : [0, T] \times \mathbf{T}^d \to \mathbf{R}^d$ with the initial value $u(0, x) = u_0(x)$ for some smooth periodic divergence-free vector field $u_0 : \mathbf{T}^d \to \mathbf{R}^d$. Take divergences on both sides of Eq.(Leray-NSE) as

$$\partial_t (\nabla \cdot u) = \nu \Delta (\nabla \cdot u),$$

which turns out to be a heat equation. Since $\nabla \cdot u$ is periodic, smooth and vanishes at 0, u is divergence-free.

Remark. Note that, here, we seem to establish the identical-zero property via the uniqueness of the heat equation. Nevertheless, it strongly relies on the periodic assumption. More generally, the most critical weapon for uniqueness, energy estimates, cannot be established in a very general case, which will cause the pathological solutions. See [Tyc35] for the very widely-known significant counterexample to the uniqueness of heat equations – Tychonoff example.

Then, if we define that $p : [0, T] \times \mathbf{T}^d \to \mathbf{R}$ as

$$p = -\Delta^{-1}(\nabla \cdot \nabla \cdot (u \otimes u)),$$

it is obvious that (u, p) is indeed smooth and solve the Eq.(NSE) together with the initial value u_0.

2.2.2 Mild solutions

We now turn into the system Eq.(Leray-NSE). The Duhamel formula hence gives the formal solution as

$$u(t) = e^{\nu t \Delta} u_0 - \int_0^t e^{\nu(t-s)\Delta} \mathbb{P}(\nabla \cdot (u(s) \otimes u(s)))ds. \tag{3}$$

The critical problem is to find an appropriate space of functions to make sense. H. Fujita and T. Kato found the proper procedure in their masterpiece [FK64].

Definition 2.1. Let $s \geq 0, T > 0$, and let $u_0 \in H^s(\mathbf{T}^d \to \mathbf{R}^d)^0$ be divergence-free and of mean zero. An H^s*-mild solution* to the NSE with initial data u_0 is a function $u : [0, T] \times \mathbf{T}^d \to \mathbf{R}^d$ in the space

$$C_t^0 H_x^s([0, T] \times \mathbf{T}^d \to \mathbf{R}^d)^0 \cap L_t^2 H_x^{s+1}([0, T] \times \mathbf{T}^d \to \mathbf{R}^d)^0$$

that obeys the Duhamel formula Eq.(3) in the sense of distributions for all $t \in [0, T]$. We say that u is a *mild solution on* $[0, T_*)$ if it is a mild solution on any $[0, T]$ for $0 < T < T_*$.

Theorem 4 (Local well-posedness of mild solutions at high regularity)**.** *Let* $s > \frac{d}{2}$*, and* $u_0 \in H^s(\mathbf{T}^d \to \mathbf{R}^d)^0$ *be divergence-free. Then, there exists a time*

$$T \gg_{d,s} \frac{\nu}{\|u_0\|_{H_x^s}^2}$$

and an H^s*-mild solution* $u : [0, T] \times \mathbf{T}^d \to \mathbf{R}^d$ *as shown in Eq.*(3)*. Furthermore, the mild solution is unique.*

Proof. The proof requires a series of tools in harmonic analysis and PDE, such as Sobolev embedding theorem and product estimation, which further requires the Littlewood-Paley projection theory. We omit it here. Readers can refer to [Tao18] for details. □

Corollary 4.1. *Mimicking the trick in ODE, we obtain the maximal Cauchy development as follows. With the above assumptions, there exists a time* $0 < T_* \leq +\infty$ *and an* H^s*-mild solution* $u : [0, T_*) \times \mathbf{T}^d \to \mathbf{R}^d$ *to Eq.*(Leray-NSE) *such that if* $T_* < \infty$*, then* $\|u(t)\|_{H^s}$ *tends to infinity as* $t \to T_*^-$*. Such* T_* *and* u *are both unique.*

Furthermore, we can strengthen it as, if $T_* < \infty$*, then* $\|u\|_{L_t^\infty L_x^\infty([0,T_*)} = \infty$*, which is exactly the blow-up phenomena.*

Corollary 4.2. *We can hence establishing the existence for smooth functions after realizing that the RHS is free of time derivative. More precisely, if* $u_0 : \mathbf{T}^d \to \mathbf{R}^d$ *is smooth and divergence-free, then there exists a unique* $0 < T_* \geq \infty$ *and a unique smooth periodic solution* (u, p) *to Eq.*(NSE) *on* $[0, T_*) \times \mathbf{T}^d$ *with normalised pressure and blow-up criterion.*

References

[FK64] H. Fujita and T. Kato. On the navier-stokes initial value problem. i. *Arch. Rational Mech. Anal.*, 16:269–315, 1964. https://doi.org/10.1007/BF00276188.

[Ler34] Jean Leray. Sur le mouvement d'un liquide visqueux emplissant l'espace. *Acta Mathematica*, 63:193–248, 1934.

[Tao18] Terence Tao. 254a, notes 1: Local well-posedness of the navier-stokes equations, 2018. https://terrytao.wordpress.com/2018/09/16/254a-notes-1-local-well-posedness-of-the-navier-stokes-equations/.

[Tyc35] A. Tychonoff. Théorèmes d'unicité pour l'équation de la chaleur. *Mat. Sb.*, 42(2):199–216, 1935. http://mi.mathnet.ru/msb6410.