Introduction to Real Harmonic Analysis, II
BMO and Hardy Spaces

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March 2022

Abstract

This article gives a quick review of basic construction of bounded mean oscillation and Hardy spaces. We especially emphasize two corresponding splendid results, John-Nirenberg inequality and duality of H^1 and BMO. Moreover, we roughly discuss the $T(1)$ theorem as a concrete application of BMO. Such series of results reveal the special but attracting properties of a fresh space to beginners of analysis.

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1 Introduction

Parallel to the progress on convergence of Fourier series we have introduced in [Hou22], in the 1960s and 1970s, another splendid achievement of harmonic analysis is the proof of the duality of BMO and Hardy spaces. This incredible connection, together with the Fefferman-Stein inequality, further reveal that controlling oscillation is the core of controlling size of functions.

BMO space was originated in F. John's studies on elasticity theory, precisely on the concept of elastic strain [Joh61] and elaborated by L. Nirenberg [JN61]. First, BMO acts as an amazing background space as a "perfect" alternative of L^∞. For instance, many singular integral operators (SIOs) are not bounded in L^∞, but sends L^∞ functions to BMO ones, such as Calderón-Zygmund operators. Then, it has better interpolation properties than L^∞. For instance, embedding theorem of Sobolev spaces shows that $L^p_{\frac{n}{p}}(\mathbb{R}^d)$ can be embedded as

$$L^p_{\frac{n}{p}}(\mathbb{R}^d) \longhookrightarrow \bigcap_{p\leq q<+\infty} L^q(\mathbb{R}^d)\cap \mathrm{BMO}\,.$$

for any $1 < p < \infty$. Moreover, it shows astonishing connections between different topics in real harmonic analysis, such as Carleson measure, Calderón-Zygmund theory. Moreover, it even has many applications in martingale [Kaz94].

Hardy spaces were introduced by F. Riesz [Rie23], in memory of G. H. Hardy [Har15]. The theory was originally established on complex analysis and harmonic functions, fulfilled with conventional techniques in classical complex analysis. However, modern harmonic analysis constructs it via atom decomposition. Indifferent from BMO spaces, Hardy spaces were naturally as a substitution of L^1 since birth.

2 Bounded Mean Oscillation

In this section, basic geometric settings are $(\mathbb{R}^d, d_\infty, \mathscr{L})$, whose basic balls are cubes $Q = Q(x, r)$ centred at $x \in \mathbb{R}^d$ and of radius r.

2.1 Definition and basic properties

Let f be a locally integrable function, i.e., in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Let $m_X f$ be the mean of f on a finite-measure subset $X \subset \mathbb{R}^n$, i.e.,

$$m_X f = \fint_X f d\mathscr{L} = \frac{1}{\mathscr{L}(X)}\int_X f d\mathscr{L}.$$

Definition 2.1. Let f be a real or complex-valued locally integrable function of $\mathbb{R}^d$. We say that f has *bounded mean oscillation* if

$$\|f\|_* := \sup_{\substack{Q\subset\mathbb{R}^d\\ \text{Q is a cube}}} \fint_Q |f - m_Q f| < +\infty.$$

The *BMO space* consists of all the functions having bounded mean oscillation, equipped with the *BMO-norm*, $\|\cdot\|_*$.

Note that the BMO-norm is not a real norm since $\|f\|_* = 0$ iff f is almost everywhere constant, so definitely, BMO space itself is naturally not a Banach space. However, BMO $/\mathbb{K}$, where $\mathbb{K}$ is the base field $\mathbb{R}$ or $\mathbb{C}$, is indeed a Banach space. Thus, the convergence of BMO mostly denotes the convergence modulo constant.

Remark. We have that $L^\infty(\mathbb{R}^d) \subset \mathrm{BMO}$ with estimation

$$\|f\|_* \leq 2\|f\|_\infty,$$

which again ensures the intuition that BMO will act as an alternative of L^∞.

Remark. For any $f \in \mathrm{BMO}$,

$$f_{t,x_0}(x) := f\left(\frac{x - x_0}{t}\right)$$

also lies in BMO with the same BMO-norm. It implies that the space is invariant of translations and scaling.

Remark. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and suppose that there exists some constant $C > 0$ such that for all cubes $Q \subset \mathbb{R}^d$, there exists a constant $c_Q \in \mathbb{K}$ such that

$$⨍_Q |f - c_Q| \leq C.$$

Then, $f \in \mathrm{BMO}$ with $\|f\|_* \leq 2C$. It implies that the space is independent of the baseline we choose.

A critical toy model will strengthen our insight of this space.

Example 1. *The function* $\log|x|$ *of* $\mathbb{R}^d$ *to* $\mathbb{R}$ *lies in* BMO*, but* $\mathbb{1}_{\mathbb{R}_+^*}(x)\log x$ *of* $\mathbb{R}$ *to* $\mathbb{R}$ *does not lie in* BMO*. It implies that* $L^\infty(\mathbb{R}^d) \subsetneq \mathrm{BMO}$ *and* BMO *is **not** stable under multiplication by characteristic functions.*

Another strange property will distinguish it with general L^p more clearly.

Example 2. *Suppose that* $f \in \mathrm{BMO}$*, and then* $|f| \in \mathrm{BMO}$ *with* $\||f|\|_* \leq 2\|f\|_*$*. However, there exists an example that* $|f| \in \mathrm{BMO}$ *but* $f \notin \mathrm{BMO}$*.*

2.2 Decay of BMO functions

Since $\mathbb{1}_Q(x)(f - m_Q f)(x)$ is in L^1, Cavalieri's principle reads that

$$\|f - m_Q f\|_1 = \int_0^{+\infty} \mathscr{L}\{x \in Q : |f(x) - m_Q f| > \lambda\} d\lambda,$$

i.e., $\mathscr{L}\{x \in Q : |f(x) - m_Q f| > \lambda\}$ is in $L^1(\lambda)$. A further discussion is about the speed of decay of BMO functions, i.e., the mode of dependence of

$$\frac{\mathscr{L}\{x \in Q : |f(x) - m_Q f| > \lambda\}}{\mathscr{L}(Q)}$$

on λ. An obvious observation comes from Chebyshev-Markov inequality that

$$\mathscr{L}\{x \in Q : |f(x) - m_Q f| > \lambda\} \leq \frac{1}{\lambda}\int_Q |f(x) - m_Q f| d\mathscr{L} \leq \mathscr{L}(Q)\frac{1}{\lambda}\|f\|_*.$$

An extraordinary result by John-Nirenberg improves the mode into exponential decay, which comes from the scaling and translation invariance of BMO-norm, a typical technique in harmonic analysis. For this, we shall mix different cubes and observe the behavior of f.

Lemma 3. *Suppose that $Q, R \subset \mathbb{R}^d$ are cubes with $Q \subset R$. Let $f \in \mathrm{BMO}$. Then,*

$$|m_Q f - m_R f| \leq \frac{\mathscr{L}(R)}{\mathscr{L}(Q)} \|f\|_*.$$

Proof. Compute that

$$|m_Q f - m_R f| \leq m_Q |f - m_R f| \leq \frac{\mathscr{L}(R)}{\mathscr{L}(Q)} \fint_R |f - m_R f|.$$

□

Corollary 3.1. *Suppose that $Q, R \subset \mathbb{R}^d$ are arbitrary cubes. Then*

$$|m_R f - m_Q f| \leq C_d \|f\|_* \rho(Q, R),$$

where

$$\rho(Q, R) = \log\left(2 + \frac{\mathscr{L}(Q)}{\mathscr{L}(R)} + \frac{\mathscr{L}(R)}{\mathscr{L}(Q)} + \frac{\operatorname{dist}(Q, R)}{(\mathscr{L}(Q) \wedge \mathscr{L}(R))^{\frac{1}{n}}}\right).$$

Proof. The proof is based on "telescoping argument" in harmonic analysis. Write that $Q = Q(x, r)$ and $R = Q(y, s)$. Find $n \in \mathbb{N}$ such that $R \subset 2^n Q = Q(x, 2^n r) =: \widehat{Q}$. Lemma 3 shows that

$$\begin{aligned}
|m_Q f - m_R f| &\leq |m_{\widehat{Q}} f - m_Q f| + |m_{\widehat{Q}} - m_R f| \\
&\leq \sum_{k=0}^{n-1} |m_{2^{k+1}Q} f - m_{2^k Q} f| + |m_{\widehat{Q}} - m_R f| \\
&\leq 2^d n \|f\|_* + |m_{\widehat{Q}} - m_R f|.
\end{aligned}$$

Find $m \in \mathbb{N}$ such that $\widehat{R} := 2^m R$ is of equivalent measure with $\widehat{Q}$, and similar computation gives that

$$|m_{\widehat{Q}} - m_R f| \leq 2^d m \|f\|_* + |m_{\widehat{Q}} f - m_{\widehat{R}} f|.$$

Find $S \subset \mathbb{R}^n$ containing both $\widehat{Q}$ and $\widehat{R}$ but with equivalent size, so the last term could be controlled by a constant merely relying on d. The parameters n, m are controlled by considering $\operatorname{dist}(Q, R)$. □

Note that the corollary improves the lemma as log-dependence, which is critical in the proof for exponential decay. Now, we are ready to state John-Nirenberg theorem.

Theorem 4 (John-Nirenberg)**.** *There exists constants $C, \alpha > 0$, merely depending on the dimension d such that for all $f \in \mathrm{BMO}$ with $\|f\|_* \neq 0$, all cubes $Q \subset \mathbb{R}^d$ and $\lambda > 0$, the inequality holds that*

$$\frac{\mathscr{L}\{x \in Q : |f(x) - m_Q f| > \lambda\}}{\mathscr{L}(Q)} \leq C \exp\left(-\alpha \frac{\lambda}{\|f\|_*}\right).$$

Proof. The proof is mainly based on Hardy-Littlewood maximal operator and the observation about oscillation of f and $|f|$. $\square$

For all $1 < p < \infty$ and $f \in L^p_{\text{loc}}(\mathbb{R}^d)$, define

$$\|f\|_{*,p} := \sup_{\substack{Q \subset \mathbb{R}^d \\ Q \text{ is a cube}}} \left(\fint_Q |f - m_Q f|^p \right)^{\frac{1}{p}},$$

and $\mathrm{BMO}_p := \{f \in L^p_{\text{loc}}(\mathbb{R}^d) : \|f\|_{*,p} < \infty\}$.

Corollary 4.1. *For all* $1 < p < \infty$, $\mathrm{BMO}_p = \mathrm{BMO}$ *with* $\|f\|_{*,p} \sim \|f\|_p$.

The corollary implies that the BMO property can be regarded as independent of measurements of L^p norms.

3 Hardy Spaces

3.1 Definition

It is well-known that L^p spaces behave badly when $0 < p < 1$. An alternative in this region of L^p spaces is Hardy space, which utilizes $1 \leq p \leq +\infty$ in a viewpoint of duality. We only concentrate on the case for $p = 1$ in this section. Fundamental geometric settings of this section is $(\mathbb{R}^d, \mu, d_2)$.

Definition 3.1. Let Q be a cube of $\mathbb{R}^d$. A measurable function $a : Q \to \mathbb{C}$ is called a *p-atom* on Q for $1 < p \leq \infty$ if it satisfies that:

i. $\operatorname{supp} a \subset Q$;

ii. $\int_Q a d\mathscr{L} = 0$;

iii. $\|a\|_p \leq \mathscr{L}(Q)^{-\frac{1}{p'}}$.

The collection of p-atoms on Q is denoted by $\mathscr{A}^p_Q$, and define $\mathscr{A}^p$ as

$$\mathscr{A}^p := \bigcup_{\substack{Q \subset \mathbb{R}^d \\ Q \text{ is a cube}}} \mathscr{A}^p_Q.$$

Remark. Obviously, condition iii. shows that $\|a\|_1 \leq 1$.

Definition 3.2. Let $1 < p \leq \infty$ and $f \in L^1(\mathbb{R}^d)$. We say that $f \in H^{1,p}$ if there exist p-atoms $\{a_i\}_{n \in \mathbb{N}}$ and $(\lambda_i) \in \ell^1(\mathbb{N})$ such that f can be expressed as

$$f = \sum_{i \in \mathbb{N}} \lambda_i a_i \tag{3.1}$$

almost everywhere. The space $H^{1,p}$ is called the Hardy space, equipped with the norm

$$\|f\|_{H^{1,p}} := \inf\{\|(\lambda_i)\|_{\ell^1} : \text{ there exists } \{a_i\}_{i \in \mathbb{N}} \in (\mathscr{A}^p)^{\mathbb{N}} \text{ such that Eq.(3.1) holds}\}.$$

Remark. Observe that, for $1 < p < r < \infty$,

$$H^{1,\infty} \hookrightarrow H^{1,r} \hookrightarrow H^{1,p} \hookrightarrow L^1(\mathbb{R}^d).$$

Proposition 5. *The Hardy space $H^{1,p}$ together with norm $\|\cdot\|_{H^{1,p}}$ is a Banach space.*

Theorem 6. *For $1 < p < \infty$, $H^{1,p} = H^{1,\infty}$ with equivalence of norms.*

It again shows that the Hardy space is independent of the L^p construction, hence ensures the legality of the definition. We use H^1 directly for the Hardy space, but maybe equipped with different norms according to the construction.

Proof. The proof is based on Calderón-Zygmund decomposition, hence omitted here. The main viewpoint is to decompose a function as the sum of its average and its oscillatory part on some region specially selected. □

3.2 Duality of H^1 and BMO

The main theorem clearly demonstrates the dual of H^1 by C. Fefferman and E. Stein.

Theorem 7 (Fefferman-Stein)**.** *The dual of H^1 is isomorphic to* BMO *with equivalent norms, i.e.,*

$$(H^1)^* \simeq \mathrm{BMO}\,.$$

An intuition of the rigorous proof is the pairing of $H^1 = H^{1,2}$ and $\mathrm{BMO} = \mathrm{BMO}_2$ with corresponding norms. For any $a \in \mathscr{A}^2$ and $b \in \mathrm{BMO}$, consider the pairing

$$P(a,b) := \int_{\mathbb{R}^d} a(x)b(x)d\mathscr{L}(x).$$

It makes sense since there is a cube $Q \subset \mathbb{R}^d$ such that $\operatorname{supp} a \subset Q$ and $m_Q a = 0$, so

$$P(a,b) = \int_Q a(b - m_Q b)d\mathscr{L} \leq \|a\|_2 \|b\|_*.$$

By linearity, the pairing is well defined for $f \in \operatorname{span}(\mathscr{A}^2)$. The main difficulty is to show the density of $\operatorname{span}(\mathscr{A}^2)$ in H^1, which is highly non-trivial and only achieved recently. A slight taste could directly show that the remaining nightmare is the expression of functions in Hardy spaces, neither unique nor constrained. The clever way is to avoid this barrier using density of L^∞ in BMO, which could be easily established.

4 Application of BMO: $T(1)$ theorem

We have introduced in [Hou22] that a Calderón-Zygmund operator is a bounded L^2 operator together with its Schwartz kernel a Calderón-Zygmund function. In general, the second condition is specially extracted to form a new space of singular operators.

Definition 4.1. Let $T : C^\infty_{\mathrm{comp}}(\mathbb{R}^d) \to \mathscr{D}'(\mathbb{R}^d)$ be linear and continuous. Then, T is said be a *singular integral operator* is its Schwartz kernel is a CZK_α-function for some $\alpha > 0$, when restricted to Δ^c. The space SIO consists of all singular integral operators.

Generally speaking, for a particular concrete operator, it is not very difficult to verify whether it is in SIO. However, it would be rather troublesome to show whether it is bounded for L^2 functions. The answer amazingly relies on BMO by G. David and J.-L. Journé in 1984 [DJ84], which undoubtedly is the most highlight achievement in harmonic analysis in 1980s. To see this, let us first remember why Calderón-Zygmund operators could only be extended as bounded L^p operators for $1 < p < \infty$ via a toy model.

Example 8. *Recall the Hilber transform as $H : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ such that*

$$H(\varphi) = \mathrm{p.\,v.}\left(\frac{1}{\pi x}\right) * \varphi.$$

Show that

$$H(\mathbb{1}_{[0,1]})(x) = -\frac{1}{\pi}\log\left|\frac{x-1}{x}\right|$$

for $x \notin [0,1]$. It directly shows that H is unbounded as L^1 operators and L^∞ operators, while H is a typical Calderón-Zygmund operator.

Theorem 9. *Any Calderón-Zygmund operator $T \in \mathrm{CZO}_\alpha$ induces an extension as a bounded operator of H^1 to $L^1(\mathbb{R}^d)$, hence inducing another extension of $L^\infty(\mathbb{R}^d)$ to* BMO.

Corollary 9.1. *Let $T \in \mathrm{CZO}_\alpha$. Then T induces a bounded operator of H^1 iff $T^\top(1) = 0$ in* BMO.

Such series of observations push us forward to David-Journé theorem.

Theorem 10 (David-Journé)**.** *Let T be in* SIO. *Then T is a bounded L^2 operator iff it has weak boundedness property and $T(1), T^\top(1) \in \mathrm{BMO}$.*

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