# Introduction to Real Harmonic Analysis, I: Interpolation and Calderón-Zygmund theory

Approach from Fourier Series

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**Abstract**

In this paper, we give a quick review of the most highlight theorems in the interpolation theory and Calderón-Zygmund theory. We connect these two important contents of modern harmonic analysis via M. Riesz's work on the $L^p$-convergence of Fourier series.

## 1 Introduction

Analysis in general is a subject focusing on properties of *functions* and *operators* of functions. In contrast to functional analysis, which always tends to distill *abstract* and *qualitative* properties of specific operations, real harmonic analysis prefers to make *concrete* and *quantitative* computations. This tells apart so-called *soft* and *hard* analysis, although, in fact, they have rather deep interactions at most time.

Harmonic analysis is generally divided into two sub-fields, *real harmonic analysis* and *abstract harmonic analysis*. Roughly speaking, real harmonic analysis concerns about behaviors of numerous tangible operators on functions of $\mathbb{R}^n$, *real* or *complex*-valued. But abstract harmonic analysis inherits algebraic properties, "symmetry" of Fourier transformation, such as translations and rotations. The main reason for this dichotomy is that real harmonic analysis highly relies on geometrical properties regarding the measure, while such fine properties could hardly be generalized to a general algebraic group. We concentrate on the real harmonic analysis in this paper.

The origin of harmonic analysis should be due to Joseph Fourier for introducing the Fourier series and Fourier transformation [Fou22]. As was pointed out by Lagrange, the convergence of Fourier series has always been standing at the most central part of Fourier analysis [Pre16]. It induced a series of celebrated works as follows.

- Pointwise convergence: J. P. G. L. Dirichlet (Dirichlet kernel, Dirichlet criterion), U. Dini (Dini-Lipschitz test), L. Fejér (Fejér kernel, Fejér's theorem on uniform convergence).

- Absolute convergence: N. Wiener (Wiener algebra, Wiener's theorem for $\frac{1}{f}$), I. Gelfand (Gelfand representation theorem for simplification).

- Norm convergence: F. Riesz (Riesz-Fischer theorem for $L^2$), M. Riesz ($L^p$ for $1 < p < \infty$), Kolmogorov (Kolmogorov's counterexample for $L^1$-norm divergence).

- Almost-everywhere convergence: A. Kolmogorov (Kolmogorov's counterexample for an $L^1$-function almost everywhere divergent FS & everywhere divergent), L. Carleson (Lusin-Carleson theorem for continuous functions, Carleson theorem for $L^2$ functions), R. Hunt ($L^p, 1 < p < \infty$), J.-P. Kahane-Y. Katznelson (any zero-measure set has a continuous function divergent everywhere on it).

- Multidimensional case: C. Fefferman.

Numerous results true for one-dimensional case are still open for multi-dimensional case.

The one establishing the very age of modern harmonic analysis is the following theorem of M. Riesz.

**Theorem 1** (M. Riesz, [Rie27])**.** *Let $f$ be a function in $L^p(\mathbb{T})$ for $1 < p < \infty$, and $S_n f$ the $n$-th sum of its Fourier series. Then, $S_n f$ converges to $f$ in $L^p(\mathbb{T})$.*

In his proof, he developed an interpolation theorem, later called Riesz-Thorin theorem, which started studies on interpolation theory. He also proved the $L^p$ extension of Hilbert transformation, which was the epilogue of the Calderón-Zygmund theory. They are the very fundamental tool and main thoughts for modern harmonic analysis, although the latter has switched its focus onto general operator theory but not only the convergence mode of Fourier series.

# 2 Interpolation theory

Let us first discuss a conventional problem from matrices. Let $A = (a_{ij})_{N\times N}$ be a matrix in $M_N(\mathbb{C})$. Equip $\mathbb{C}^N$ with $\ell^p(\mathbb{C}^N)$ norms and $M_N(\mathbb{C})$ with the corresponding operator norms for $1 \leq p \leq \infty$. It is obvious that

$$\|A\|_\infty = \sup_{1\leq i\leq N} \sum_{j=1}^N |a_{ij}|, \quad \|A\|_1 = \sup_{1\leq j\leq N} \sum_{i=1}^N |a_{ij}|.$$

A natural problem is to ask what it is for the general case? Unfortunately, there is no precise formula for the general $\|A\|_p$, but instead, we have the inequality.

**Proposition 2** (Schur's lemma)**.** *Inheriting the above assumptions, it follows that*

$$\|A\|_p \leq \|A\|_1^{\frac{1}{p}} \|A\|_\infty^{1-\frac{1}{p}}.$$

The highlight of this lemma is that it reveals the general type for interpolation, the log-convexity. Note that $p$ can be written as

$$\frac{1}{p} = \frac{1/p}{1} + \frac{1-1/p}{\infty},$$

which is exactly the fraction generalisation of Hölder's type.

Another obvious but important observation is numerical interpolation. Suppose that $A, B, C$ are three positive numbers with $C \leq A, B$, and then, for any $\theta \in [0,1]$,

$$C \leq A^\theta B^{1-\theta}.$$

Let us get down to the complicated theory first by introducing its necessary language. Without specially mentioned, $(M,\mu), (N,\nu)$ are two $\sigma$-finite measure spaces, $\mathscr{D}_M$ is the space of simple, integrable functions on $M$, and $\mathscr{F}_N$ is the space of measurable functions on $N$, both valued in $\mathbb{C}$.

**Definition 2.1.** For $0 < p < \infty$, define $L^{p,\infty}(M,\mu)$ as the space consisting of $\mu$-measurable functions $f$ with

$$\sup_{\lambda > 0} \lambda^p \mu\{x \in M : |f(x)| > \lambda\} < \infty,$$

hence equipped with the norm

$$\|f\|_{p,\infty} = \left( \sup_{\lambda>0} \lambda^p \mu\{x \in M : |f(x)| > \lambda\} \right)^{\frac{1}{p}}.$$

For $p = \infty$, define $L^{\infty,\infty}(M,\mu)$ as $L^\infty(M,\mu)$ for coherence.

Differences between $L^{p,\infty}$ spaces and $L^p$ spaces can be directly shown via Cavalieri's principle.

**Proposition 3.** *Let $f$ be a function in $L^p(M,\mu)$ for $0 < p < \infty$. Then*

$$\int_M |f|^p d\mu = p \int_0^\infty \lambda^{p-1} \mu\{x \in M : |f(x)| > \lambda\} d\lambda.$$

The next definitions are about properties of operators.

**Definition 2.2** (Sublinear operator)**.** We say that $T : \mathscr{D}_M \to \mathscr{F}_N$ is *sublinear* if for any $f_1, f_2 \in \mathscr{D}_M$ and any $x \in M$,

$$|T(f_1 + f_2)(x)| \leq |Tf_1(x)| + |Tf_2(x)|.$$

**Definition 2.3** (Strong type, weak type)**.** Let $1 \leq p, q \leq \infty$ be two real numbers. We say a sublinear operator $T : \mathscr{D}_M \to \mathscr{F}_N$ is of *strong type* $(p,q)$ if $T$ induces a bounded lienar operator from $L^p(M,\mu)$ to $L^q(N,\nu)$, and $T$ is of *weak type* $(p,q)$ if $T$ induces a bounded lienar operator from $L^p(M,\mu)$ to $L^{q,\infty}(N,\nu)$.

Riesz-Thorin interpolation theorem reveals the complex interpolation with log-convexity.

**Theorem 4** (Riesz-Thorin)**.** *Let $T : \mathscr{D}_M \to \mathscr{F}_N$ be a $\mathbb{C}$-linear operator. Let $1 \le p_1, p_2, r_1, r_2 \le \infty$ be such that $T$ is of strong type $(p_1, p_2)$ and $(r_1, r_2)$ with bounds $A, B$, respectively. Then, $T$ is of strong type $(q_1, q_2)$ for any with bound $C$ satisfying*

$$C \le A^{1-\theta} B^{\theta},$$

*where*

$$\frac{1}{q_1} = \frac{1-\theta}{p_1} + \frac{\theta}{r_1}, \quad \frac{1}{q_2} = \frac{1-\theta}{p_2} + \frac{\theta}{r_2}$$

*for some $\theta \in [0, 1]$.*

*Proof.* The proof utilizes Hadamard's three-line theorem and dual argument. $\square$

However, the endpoint hypothesis does not work well with numerous important operators in harmonic analysis, such as Hardy-Littlewood maximal operator. It is not $\mathbb{C}$-linear, and merely of weak type (1,1), but can be extended to be of any strong type $(p, p)$ for $1 < p \le \infty$. Such phenomenon enlightens the following real interpolation theorem.

**Theorem 5** (Marcinkiewicz)**.** *Further assume that $\mathscr{D}_M$ is stable under multiplication by characteristic functions. Suppose that $1 \le p_1 < p_2 \le \infty, 1 \le q_1 < q_2 \le \infty$ with $p_1 \le q_1, p_2 \le q_2$. Let $T : \mathscr{D}_M \to \mathscr{F}_N$ be a $\mathbb{C}$-sublinear operator of weak type $(p_1, q_1)$ and $(p_2, q_2)$. Then, for all $p \in (p_1, p_2)$, $T$ is of strong type $(p, q)$ with*

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \quad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$$

*for some $\theta \in (0, 1)$.*

These two theorems can be explicitly shown via the following diagram, where the horizontal axis is $\frac{1}{p}$, and the vertical is $\frac{1}{q}$.

# 3 Calderón-Zygmund theory

Another significant perspective of M. Riesz's proof of Theorem 1 is the $L^p$-boundedness of Hilbert transformation, which started the era of a splendid theory of singular integrals.

**Definition 3.1** (Hilbert transformation)**.** Let $H : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ be as

$$H(\varphi) = \mathrm{p.\,v.} \left( \frac{1}{\pi x} \right) * \varphi.$$

Via Fourier transformation, one may see that

$$H(\varphi)(x) = \mathscr{F}^{-1}[-i \operatorname{sgn}(\xi) \widehat{\varphi}(\xi)](x),$$

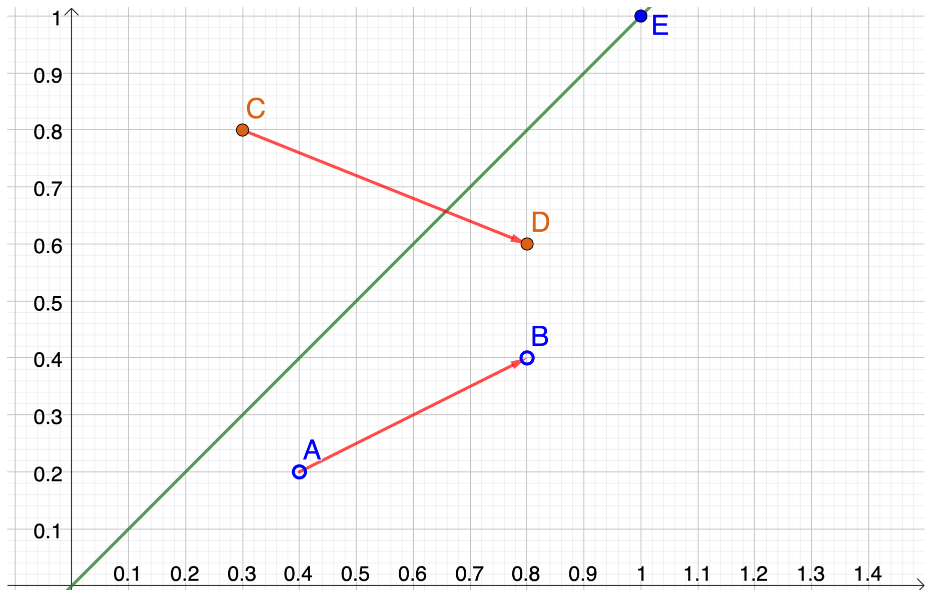


Figure 1: Interpolation Diagram

i.e., it can be regarded as the form

$$H(\varphi)(x) = \int_{\mathbb{R}} -i\,\mathrm{sgn}(\xi)\widehat{\varphi}(\xi)e^{ix\xi}d\xi = \int_{\mathbb{R}\times\mathbb{R}} e^{i\xi(x-y)}[-i\,\mathrm{sgn}(\xi)]\varphi(y)d\xi dy. \tag{1}$$

To deal with such oscillatory integrals, analysts are divided into two parts. One turned to strengthen the differentiable and **local** properties of functions outside of oscillation part $e^{i\xi(x-y)}$, which turns to be *microlocal analysis*. Main topics in microlocal analysis is to use *symbols*, in particular, $a(x,\xi) = -i\,\mathrm{sgn}(\xi)$ in our example, to reveal operations of operators, propagation of singularities, etc. The other stood still in the **global** analysis of such operators and developed the Calderón-Zygmund theory. A seminal global theorem on such Fourier multipliers is Hörmander-Mikhlin theorem.

**Theorem 6** (Hörmander-Mikhlin)**.** *Let $m \in L^\infty(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus \{0\})$ be such that for any $\alpha \in \mathbb{N}^d$, there exists $C_\alpha < \infty$ such that*

$$\left|\frac{\partial^{|\alpha|}}{\partial \xi^\alpha} m(\xi)\right| \le \frac{C_\alpha}{|\xi|^{|\alpha|}}$$

*for any $\xi \in \mathbb{R}^d \setminus \{0\}$. Then, the operator*

$$T_m = \mathscr{F}^{-1}\left(m\widehat{f}\right)$$

*is of strong type $(p,p)$ for $p \in (1,\infty)$.*

Such operator is called of *Hörmander-Mikhlin type* or HM type, and it is trivial that Hilbert transformation is of this type.

Instead of dividing the integrand as oscillation part and magnitude part in Eq.(1), A. Calderón and A. Zygmund inherited the ancient perspective of kernels by Schwartz kernel theorem, and they refilled a new kind of operators. Let $\Delta$ be the diagonal of $\mathbb{R}^d$, $\{(x,x) \in \mathbb{R}^n \times \mathbb{R}^d : x \in \mathbb{R}^d\}$.

**Definition 3.2** (Calderón-Zygmund kernel)**.** For $0 < \alpha \leq 1$, a *Calderón-Zygmund kernel of order* $\alpha$ is a continuous function $K : \Delta^C \to \mathbb{K}$ such that there exists some $C > 0$ satisfying the followings.

i. (Size) For any $(x,y) \in \Delta^C$,

$$|K(x,y)| \leq \frac{C}{|x-y|^n}.$$

ii. (Controlled oscillation on $x$) For any $(x,y),(x',y) \in \mathbb{R}^n$ with $|x-x'| \leq \frac{1}{2}|x-y|$ but $x \neq y$, it holds that

$$|K(x,y) - K(x',y)| \leq \frac{C}{|x-y|^d}\left(\frac{|x-x'|}{|x-y|}\right)^\alpha.$$

iii. (Controlled oscillation on $y$) For any $(x,y),(x,y') \in \mathbb{R}^n$ with $|y-y'| \leq \frac{1}{2}|x-y|$ but $x \neq y$, it holds that

$$|K(x,y) - K(x,y')| \leq \frac{C}{|x-y|^d}\left(\frac{|y-y'|}{|x-y|}\right)^\alpha.$$

The space $\mathrm{CZK}_\alpha$ consists of Calderón-Zygmund kernels of order $\alpha$, and is equipped with the norm

$$\|K\|_\alpha = \inf\{C \in \mathbb{R}_+ : \text{i.-iii. holds for } C\}.$$

**Definition 3.3** (Calderón-Zygmund operator)**.** For $0 < \alpha \leq 1$, a *Calderón-Zygmund operator of order* $\alpha$ is a linear operator $T \in \mathcal{L}(L^2(\mathbb{R}^d))$ associated to a kernel $K \in \mathrm{CZK}_\alpha$, that is, for any $f \in L^2(\mathbb{R}^d)$ with compact support, it holds that

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy$$

for a.e. $x \in (\operatorname{supp} f)^C$. The space $\mathrm{CZO}_\alpha$ consists of Calderón-Zygmund operators of order $\alpha$ and is equipped with the norm

$$\|T\|_{\mathrm{CZO}_\alpha} := \|K\|_\alpha + \|T\|_2.$$

**Theorem 7.** *Any operator in* $\mathrm{CZO}_\alpha$ *has a bounded extension of* $L^p(\mathbb{R}^d)$ *for any* $1 < p < \infty$.

*Proof.* The most difficult part is to show that $T$ can be extended as an operator $\widetilde{T}$ of weak type $(1, 1)$. Supposing so, via the Marcinkiewicz interpolation theorem, $\widetilde{T}$ has strong type $(p, p)$ for any $1 < p \leq 2$. The other side of extension can be achieved via the dual argument.

The extension of weak type (1,1) is established on the Calderón-Zygmund decomposition, which was derived from an amazing perspective of bounded-mean-oscillation functions. □

## References


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