Boundedness of operators on tent spaces and applications to PDE's

Continuité d'opérateurs sur les espaces de tentes et applications aux EDP

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Résumé: Cette thèse étudie la continuité des opérateurs d'intégrale singulière sur les espaces de tentes et explore leurs applications aux EDP paraboliques. Nous développons un cadre théorique fondé sur la décroissance $L^p - L^q$ hors-diagonale. Ce cadre fournit un résultat amélioré (éventuellement optimal) concernant la continuité de l'opérateur de régularité maximale sur ces espaces.

En nous appuyant sur ce cadre, nous établissons un panorama complet concernant l'existence, l'unicité et la représentation des solutions faibles aux problèmes de Cauchy paraboliques non-autonomes sous forme divergence. Les coefficients sont supposés être uniformément elliptiques, bornés, mesurables et possiblement à valeurs complexes, sans hypothèses supplémentaires de régularité ou de symétrie. Les données initiales sont des distributions tempérées prises dans les espaces Hardy–Sobolev homogènes $\dot{H}^{s,p}$, tandis que les termes sources appartiennent à certaines échelles d'espaces de tentes pondérés. Les solutions faibles sont construites de manière à ce que leurs gradients appartiennent aux espaces de tentes pondérés $T^p_{s/2}$.

Nous appliquons enfin les espaces de tentes à l'équation de Navier–Stokes, en résovant un problème ouvert concernant la continuité temporelle et le comportement en grand temps des solutions mild dans l'espace de Koch–Tataru évoluant à partir de données initiales dans BMO^{-1}.

Title: Boundedness of operators on tent spaces and applications to PDE's

Keywords: Tent spaces, singular integral operators, well-posedness of parabolic Cauchy problems, heat equation, Navier–Stokes equations, homogeneous Hardy–Sobolev spaces.

Abstract: This thesis investigates boundedness of singular integral operators on tent spaces and explores their applications to parabolic PDE's. We develop a framework based on the $L^p - L^q$ off-diagonal decay, which provides us with an enhanced (possibly optimal) boundedness result for the maximal regularity operator on tent spaces.

Using this framework, we establish a complete picture for existence, uniqueness, and representation of weak solutions to non-autonomous parabolic Cauchy problems of divergence type. The coefficients are assumed to be uniformly elliptic, bounded, measurable, and possibly complex-valued, without any additional regularity or symmetry conditions. The initial data are tempered distributions taken in homogeneous Hardy–Sobolev spaces $\dot{H}^{s,p}$, and source terms belong to certain scales of weighted tent spaces. Weak solutions are constructed with their gradients in the weighted tent spaces $T^p_{s/2}$.

We also apply tent space theory to the Navier–Stokes equation, solving a long-standing open problem concerning the time continuity and long-time behavior of mild solutions in the Koch–Tataru space evolving from initial data in BMO^{-1}.

To my parents and my grandparents

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And to the Moon, for her quiet company, night after night.

Chapter 1

Introduction

“来如流水兮逝如风，
不知所来兮何所终。”[1]

改编自峨默《鲁拜集》第28首

1.1 Main objective

The main objective of this thesis is to investigate boundedness of singular integral operators on tent spaces and to explore the applications to parabolic partial differential equations, including both the linear parabolic Cauchy problems of divergence type and the non-linear incompressible Navier–Stokes equation.

The first part of this thesis is devoted to establishing a theory of singular integral operators on tent spaces based on the notion of $L^p - L^q$ off-diagonal decay. We extend and improve the pioneering work of Auscher *et al.* [AKMP12]; for instance, our theory includes more operators of particular interest in the study of (parabolic) PDE's, such as the Duhamel operator and its gradient, and the maximal regularity operator. We obtain a nearly optimal boundedness result for these operators on tent spaces.

In the second part of this thesis, we establish an operator-theoretic framework for the study of parabolic Cauchy problems of divergence type

$$\begin{cases} \partial_t u - \mathrm{div}_x(A(t,x)\nabla_x u) = f + \mathrm{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases}, \tag{1.1}$$

by using the theory of singular integral operators on tent spaces. We present a complete picture for existence, uniqueness, and representation of (energy)

[1]Adapted from *Rubáiyát of Omar Khayyám* XXVIII. English translation by Edward FitzGerald: “*I came like Water, and like Wind I go.*”

weak solutions to (1.1). The complex-valued coefficient matrix $A(t,x)$ is assumed to be uniformly elliptic, but merely bounded and measurable in both space and time. Specifically, $A \in L^\infty((0,\infty)\times\mathbb{R}^n;\mathrm{Mat}_n(\mathbb{C}))$ is called *uniformly elliptic*, if there exist $\Lambda_0,\Lambda_1>0$ so that for a.e. $(t,x)\in(0,\infty)\times\mathbb{R}^n$ and any $\xi,\eta\in\mathbb{C}^n$,

$$\Re(\langle A(t,x)\xi,\xi\rangle) \geq \Lambda_0|\xi|^2, \quad |\langle A(t,x)\xi,\eta\rangle| \leq \Lambda_1|\xi||\eta|. \tag{1.2}$$

It is worth mentioning that we do *not* impose *any* assumptions on symmetry or regularity of the coefficients in either the time variable t or the space variable x.

By the initial condition $u(0)=u_0$, we require that $u(t)$ tends to 0 as $t\to 0$ in distributional sense, *i.e.*, in $\mathscr{D}'(\mathbb{R}^n)$.

This model serves as a simplified but representative example for parabolic systems. In fact, our results presented herein readily extend to parabolic systems where the ellipticity condition is replaced by a Gårding inequality on $\mathbb{R}^n$ uniformly with respect to t. That is, there exists $\Lambda_0>0$ so that for a.e. $t>0$ and any $\varphi\in\dot{H}^1(\mathbb{R}^n;\mathbb{C}^m)$,

$$\Re\int_{\mathbb{R}^n}(A(t,x)\nabla\varphi(x))\cdot\overline{\nabla\varphi}(x) \geq \Lambda_0\|\nabla\varphi\|^2_{L^2(\mathbb{R}^n)}.$$

Here, $A(t,x)=(A^{\alpha,\beta}_{i,j}(t,x))^{1\leq\alpha,\beta\leq n}_{1\leq i,j\leq m}$ belongs to $L^\infty((0,\infty)\times\mathbb{R}^n;\mathrm{Mat}_{mn}(\mathbb{C}))$, and we use the notation

$$(A(t,x)\nabla\varphi(x))\cdot\overline{\nabla\varphi}(x) := \sum_{\substack{1\leq i,j\leq m\\1\leq\alpha,\beta\leq n}} A^{\alpha,\beta}_{i,j}(t,x)\partial_\beta\varphi^j(x)\overline{\partial_\alpha\varphi^i}(x).$$

The homogeneous Sobolev space $\dot{H}^1(\mathbb{R}^n;\mathbb{C}^m)$ is the closure of $C^\infty_{\mathsf{c}}(\mathbb{R}^n;\mathbb{C}^m)$ with respect to the semi-norm $\|\nabla\varphi\|_{L^2(\mathbb{R}^n;\mathbb{C}^m)}$.

The third part of this thesis is concerned with the incompressible Navier–Stokes equation. By using the tent space theory, we solve a long-standing open problem concerning the time continuity and long-time behavior of mild solutions in the Koch–Tataru space evolving from initial data in the endpoint critical space BMO^{-1}.

1.2 State of the art

In order to have a better understanding of the role of tent spaces, we first give a brief review of two classical theories for well-posedness of linear parabolic Cauchy problems of type (1.1): the L^p-maximal regularity theory and Lions' maximal regularity theory. Then we introduce the theory using tent spaces.

1.2.1 L^p-maximal regularity

This theory is mainly specialized for the case $F = 0$. Let us begin with the autonomous case $A = A(x)$, *i.e.*, the coefficients are time-independent. Define the operator

$$L := -\operatorname{div}(A(x)\nabla) \tag{1.3}$$

on $L^2(\mathbb{R}^n)$ with domain

$$D(L) := \{\varphi \in L^2(\mathbb{R}^n) : \operatorname{div}(A(x)\nabla\varphi) \in L^2(\mathbb{R}^n)\}.$$

Thanks to the theory of maximal accretive operators, we know that $-L$ generates an analytic semigroup on the Banach space $X = L^2(\mathbb{R}^n)$, denoted by $(e^{-tL})_{t\geq 0}$. One may thus interpret the Cauchy problem (1.1) as an abstract Cauchy problem on the Banach space X of the form

$$\begin{cases} \partial_t u + Lu = f & \text{in } X, \quad 0 < t < T \\ u(0) = 0 \end{cases}, \tag{1.4}$$

and adopt the notion of *mild solutions*. More precisely, for $0 < T < \infty$ and $f \in L^1((0,T);X)$, we say $u \in L^1((0,T);X)$ is a *mild solution* to (1.4), if for any $t \in [0,T]$ and $x^* \in D(L^*) \subset X^*$,

$$\langle x^*, u(t)\rangle + \int_0^t \langle L^*x^*, u(s)\rangle ds = \int_0^t \langle x^*, f(s)\rangle ds.$$

The pairs are understood as the action of the dual space X^* on X. Indeed, one can show that there exists a unique mild solution to (1.4), given by the *Duhamel operator*

$$\mathcal{L}_1(f)(t) := \int_0^t e^{-(t-s)L} f(s) ds, \quad t > 0. \tag{1.5}$$

The integrals in (1.5) make sense as Bochner integrals valued in X.

To accommodate initial data, one may seek an exponent $p \in (1,\infty)$ so that for any $f \in L^p((0,T);X)$, the mild solution u (given by the Duhamel operator) satisfies the *L^p-maximal regularity estimates*

$$\|\partial_t u\|_{L^p((0,T);X)} + \|Lu\|_{L^p((0,T);X)} \lesssim \|f\|_{L^p((0,T);X)}. \tag{1.6}$$

If it holds, then u is continuous in time, valued in the real interpolation space $Y = (D(L), X)_{1/p,p}$. Hence, one may take initial data $u_0 \in Y$.

This strategy is grounded in a general operator-theoretic framework and applies to any generator $-L$ of an analytic semigroup on a Banach space X (see, for instance, the treatise of Ladyženskaja, Solonnikov and Ural'ceva [LSU68]). One approach to establishing the estimate (1.6) for $1 < p <$

∞ is to show that the *maximal regularity operator* $\mathcal{L}_0$, initially defined on $L^2((0,T);D(L))$ by

$$\mathcal{L}_0(f)(t) := \int_0^t Le^{-(t-s)L} f(s)ds, \quad t > 0, \tag{1.7}$$

extends to a bounded operator on $L^p((0,T);X)$. Observe that it is a singular integral operator since the operator norm $\|Le^{-(t-s)L}\|_{\mathscr{L}(L^2(\mathbb{R}^n))}$ is in the order of $|t-s|^{-1}$. Therefore, techniques from harmonic analysis become pertinent. For Hilbert spaces $X = \mathfrak{H}$, De Simon [dS64] first used Fourier multipliers to prove that given any generator $-L$ of an analytic semigroup on $\mathfrak{H}$, the estimate (1.6) holds for all $p \in (1,\infty)$. In a more general context of UMD Banach spaces X, the characterization of L^p-maximal regularity was obtained by Weis [Wei01] using R-boundedness of the semigroup (this notion was originated from the work of Bourgain [Bou86]). We refer the reader to the survey [KW04] for a comprehensive overview.

For the non-autonomous case (time-dependent coefficients), one may also define $L(t) := -\operatorname{div}(A(t)\nabla)$ as an operator on X for a.e. $t > 0$ and desire the estimate (1.6). However, the characterization of L^p-maximal regularity for time-dependent coefficients $A(t,x)$ remains widely open. Existing affirmative results require additional Sobolev regularity assumptions, even in the simplest case $X = L^2(\mathbb{R}^n)$ and $p = 2$, see [DZ17, Fac18, AO19]. The reader is also referred to the book [HvNVW23, Chapter 17] for the state of the art.

Besides, there is a very limited range of the exponents q near 2 for which the theory can apply to the Banach space $X = L^q(\mathbb{R}^n)$ using R-boundedness, see [BK03]. Indeed, it has been shown in [HMM11] that for any $q < \frac{2n}{n+2}$ (resp. $r > \frac{2n}{n-2}$), there exists a *time-independent* matrix $A = A(x)$ so that the semigroup (e^{-tL}) is not necessarily bounded on $L^q(\mathbb{R}^n)$ (resp. $L^r(\mathbb{R}^n)$).

Third, as aforementioned, the initial data compatible with this theory must be taken from the real interpolation space $Y = (D(L), X)_{1/p,p}$. For the Laplacian $L = -\Delta$, given $X = L^q(\mathbb{R}^n)$, then Y identifies with the inhomogeneous Besov space $B^{2(1-1/p)}_{q,p}$. This excludes rough initial data taken in many (homogeneous) function (or distribution) spaces of particular interest in PDE's, even the Lebesgue spaces.

1.2.2 Lions' maximal regularity

The second theory is based on the notion of (energy) weak solutions. We say $u \in L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}}(\mathbb{R}^n))$ is a *weak solution* to the equation

$$\partial_t u - \operatorname{div}(A(t,x)\nabla u) = f + \operatorname{div} F,$$

if for any $\phi \in C^\infty_{\mathsf{c}}((0,\infty) \times \mathbb{R}^n)$,

$$-\int_{(0,\infty)\times\mathbb{R}^n} u\, \partial_t \phi + \int_{(0,\infty)\times\mathbb{R}^n} (A(t,x)\nabla u) \cdot \nabla \phi = (f,\phi) - (F, \nabla\phi).$$

The pairs on the right-hand side denote pairing of distributions and test functions on $(0,\infty)\times\mathbb{R}^n$. By the initial condition $u(0)=u_0$, we require that $u(t)$ converges to u_0 as $t\to 0$ in $\mathscr{D}'(\mathbb{R}^n)$.

It was J.-L. Lions [Lio57] who first established the L^2-theory of weak solutions. More precisely, for $0<T<\infty$, $u_0\in L^2(\mathbb{R}^n)$, $f=0$, and $F\in L^2((0,T)\times\mathbb{R}^n)$, there exists a weak solution u in the energy space

$$L^\infty((0,T);L^2(\mathbb{R}^n))\cap L^2((0,T);H^1(\mathbb{R}^n)) \tag{1.8}$$

to the Cauchy problem

$$\begin{cases}\partial_t u-\operatorname{div}(A(t,x)\nabla u)=\operatorname{div}F, & 0<t<T,\ x\in\mathbb{R}^n\\ u(0)=u_0\end{cases},$$

which also satisfies the energy inequality

$$\sup_{0\le t\le T}\|u(t)\|_{L^2(\mathbb{R}^n)}+\|\nabla u\|_{L^2((0,T)\times\mathbb{R}^n)}\lesssim\|u_0\|_{L^2(\mathbb{R}^n)}+\|F\|_{L^2((0,T)\times\mathbb{R}^n)}.$$

Notably, there is no assumption needed on regularity or symmetry of the coefficients, and it is compatible with rough initial data in $L^2(\mathbb{R}^n)$.

Lions also proved that the space

$$\left\{u\in L^2((0,T);H^1(\mathbb{R}^n)):\partial_t u\in L^2((0,T);H^{-1}(\mathbb{R}^n))\right\}$$

embeds into $C([0,T];L^2(\mathbb{R}^n))$ (known as *Lions' embedding theorem*). Therefore, any weak solution in the energy space to the equation $\partial_t u-\operatorname{div}(A\nabla u)=0$ processes a trace u_0 in $L^2(\mathbb{R}^n)$.

For real coefficients, much more is known since the maximum principle becomes available. Indeed, this leads to local Hölder regularity of weak solutions, as demonstrated in the renowned De Giorgi–Nash–Moser theory [DG57, Nas58, Mos61, Mos64]. For real coefficients, the complete L^2-theory was due to Aronson [Aro68, Aro67], who constructed fundamental solutions $K(t,s;x,y)$ and obtained that for any $u_0\in L^2(\mathbb{R}^n)$, the integral

$$u(t,x)=\int_{\mathbb{R}^n}K(t,s;x,y)u_0(y)dy$$

represents the unique weak solution to the homogeneous Cauchy problem

$$\begin{cases}\partial_t u-\operatorname{div}(A(t,x)\nabla u)=0, & s<t<T\\ u(s)=u_0\end{cases}$$

in the energy space (1.8) asserted by Lions' L^2-theory. Moreover, $K(t,s;x,y)$ satisfies pointwise Gaussian decay

$$0\le K(t,s;x,y)\lesssim\frac{1}{(t-s)^{n/2}}\exp\left(-c\frac{|x-y|^2}{t-s}\right) \tag{1.9}$$

for some $c > 0$. For $0 \leq s < t < T$, the operator $\Gamma_A(t,s)$ defined on $L^2(\mathbb{R}^n)$ by

$$(\Gamma_A(t,s)u_0)(x) := \int_{\mathbb{R}^n} K(t,s;x,y)u_0(y)dy, \quad x \in \mathbb{R}^n$$

is called the *propagator associated with* A. One can show that $\Gamma_A(t,s)$ is a contraction on $L^2(\mathbb{R}^n)$, using the energy inequality.

Aronson also considered initial data with growth in the order of the inverse of a Gaussian, *i.e.*, $e^{-\alpha|x|^2}u_0 \in L^2(\mathbb{R}^n)$ for $\alpha \geq 0$. The solution class is given by the condition

$$\int_0^T \int_{\mathbb{R}^n} e^{-\gamma|x|^2}|u(t,x)|^2 dtdx < \infty, \tag{1.10}$$

where $\gamma > 0$ is a constant determined by A and α. Source terms with growth in the order of the inverse of a Gaussian were also considered.

These results do yield well-posedness of weak solutions with initial data $u_0 \in L^p(\mathbb{R}^n)$ for $2 \leq p \leq \infty$. However, in contrast to the results of Lions, one can not generally expect to recover traces in $L^p(\mathbb{R}^n)$ (and consequently, the representation of solutions) from Aronson's solution class, as it is too broad.

This limitation suggests to derive finer *a priori* estimates, for instance, the mixed $L^p_t L^q_x$-estimates for weak solutions and their derivatives, particularly the highest order derivative ∇u. [2] However, to the best of our knowledge, obtaining $L^p_t L^q_x$-estimates for arbitrary coefficient matrices $A(t,x)$ remains an open problem. Partial results exist under additional assumptions. For instance, when $x \mapsto A(t,x)$ is uniformly continuous, the problem has been addressed by Ladyženskaja, Solonnikov and Ural'ceva [LSU68]. Krylov [Kry07b] later relaxed this condition, requiring only that $x \mapsto A(t,x)$ belongs to VMO. These results remain valid for complex coefficients and even for parabolic systems, see the works of Dong and Kim [DK11a, DK11b, DK18].

The fundamental strategy underlying these results is to first derive the estimates for simplified cases, such as constant or space-independent coefficients, before extending them to more general ones via perturbation arguments, for which the regularity assumptions on coefficients seem to be unavoidable.

Besides, for complex coefficients, both the maximum principle and the Gaussian decay of the propagators (cf. (1.9)) fail, even in the autonomous case, see [ACT96, AT98].

1.2.3 Tent spaces

Let us introduce a possible alternative theory by using *(weighted) tent spaces*, which leads to finer estimates for rough coefficients and rough initial data.

[2] For (energy) weak solutions u to parabolic equations of divergence type, the second order derivatives $\nabla^2 u$ and $\mathrm{div}(A\nabla u)$ could only be understood as distributions, as well as $\partial_t u$.

Tent spaces were introduced by Coifman, Meyer and Stein [CMS85] in 1985, and originated from the work of Fefferman and Stein [FS72] on real Hardy spaces in 1972. These spaces provide a unified framework for studying many function spaces on $\mathbb{R}^n$ via their extensions to the upper-half space $(0,\infty)\times\mathbb{R}^n$. As we shall see, these spaces include L^p-spaces, real Hardy spaces, and the John–Nirenberg bounded mean oscillation space BMO.

Let us recall the definition of tent spaces (in the parabolic settings). For $\beta \in \mathbb{R}$ and $0 < p < \infty$, the *(parabolic) tent space* T^p_β consists of measurable functions $u(t,y)$ on the upper-half space $(0,\infty)\times\mathbb{R}^n$ for which

$$\|u\|_{T^p_\beta} := \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \fint_{B(x,t^{1/2})} |t^{-\beta} u(t,y)|^2 dy dt \right)^{p/2} dx \right)^{1/p} < \infty.$$

The form of the inner integral originates from *Lusin's area functional*, and takes the name of *conical square function*. For $p=\infty$, the norm is given by the *Carleson functional* as

$$\|u\|_{T^\infty_\beta} := \sup_{B\colon \text{balls in } \mathbb{R}^n} \left(\frac{1}{|B|} \int_0^{r(B)^2} \int_B |t^{-\beta} u(t,y)|^2 dy dt \right)^{1/2}.$$

As we shall see below, the exponent β in time is related to regularity.

The key insight is that their norms first use localized L^2 integrability in the interior, then supplemented by a non-tangential control to access the limit at the boundary (materialized as $\mathbb{R}^n$), especially where standard mixed $L^p_t L^q_x$-spaces fall short. This adaptability makes tent spaces particularly better suited to the analysis of boundary value problems or initial value problems.

Additionally, the interior localization is ideal for studying boundedness of the operators that exhibit localized L^2 decay but no pointwise decay. A prototypical example of such operators is exactly the semigroup $(e^{-tL})_{t\geq 0}$ generated by the elliptic operator $-L = \operatorname{div}(A(x)\nabla)$, as defined in (1.3), with rough (only bounded and measurable) complex coefficients.

These properties have thus driven remarkable advancements during the past two decades in applications of tent spaces (and its variants) towards PDE's, especially elliptic equations with rough coefficients or rough boundary values. For instance, Auscher and Egert [AE23a] extensively employ this fertile framework to establish well-posedness of Dirichlet/regularity/Neumann boundary value problems of elliptic systems with block structure, addressing the largest possible spaces for boundary values. No regularity or symmetry conditions on coefficients were imposed in their work.

1.2.4 Tent spaces encounter parabolic equations

To the best of our knowledge, the very first use of tent spaces (in the form of Carleson measures) to solve evolution equations is due to Koch and Tataru

[KT01] on global well-posedness of the incompressible Navier–Stokes equation with small initial data in the endpoint critical space BMO^{-1}, which remains optimal to date.

For the study of parabolic Cauchy problems of divergence type (1.1), the pioneering works of Auscher, Monniaux and Portal [AMP12] and Kriegler [AKMP12] first investigated bounded extension of the maximal regularity operator $\mathcal{L}_0$ (for time-independent coefficients) to weighted tent spaces T^p_β. A further step has been taken in [AMP19], where they established well-posedness of (energy) weak solutions (within Lions' maximal regularity theory) to the non-autonomous *homogeneous Cauchy problem*

$$\begin{cases} \partial_t u - \operatorname{div}(A(t,x)\nabla u) = 0 \\ u(0) = u_0 \end{cases}. \tag{HC}$$

The initial data u_0 can be taken in $L^p(\mathbb{R}^n)$ for $2-\epsilon<p<\infty$, where $\epsilon>0$ is a constant depending on the ellipticity of A. For real coefficients, ϵ can be taken as 1. Still, the coefficient matrix $A(t,x)$ is only assumed to be uniformly elliptic, bounded, measurable, and complex-valued.

To this end, they first reexamine Lions' L^2-theory for (HC) and extend it to $T=\infty$ within the solution class $\nabla u \in L^2(\mathbb{R}^{1+n}_+)$. By shifting the time, they define the *propagators* $(\Gamma_A(t,s))_{0\le s\le t<\infty}$ as a family of contractions on $L^2(\mathbb{R}^n)$ so that for any $s\ge 0$ and $h\in L^2(\mathbb{R}^n)$,

$$u(t,x) := (\Gamma_A(t,s)h)(x) \tag{1.11}$$

is the unique weak solution on $(s,\infty)\times\mathbb{R}^n$ to the Cauchy problem

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(t,x)\nabla_x u) = 0, \quad (t,x)\in(s,\infty)\times\mathbb{R}^n \\ u(s) = h \end{cases}$$

with $\nabla u\in L^2((s,\infty)\times\mathbb{R}^n)$. This definition agrees with Aronson's definition by fundamental solutions for real coefficients but there might not be a representation with a kernel.

Then the weak solution u is constructed by extension of the *propagator solution map* $\mathcal{E}_A$, initially defined from $L^2(\mathbb{R}^n)$ to $L^2_{\mathsf{loc}}((0,\infty);W^{1,2}_{\mathsf{loc}}(\mathbb{R}^n))$ by

$$\mathcal{E}_A(u_0)(t,x) := (\Gamma_A(t,0)u_0)(x), \quad t>0,\ x\in\mathbb{R}^n. \tag{1.12}$$

The solution class is characterized by the tent space T^p_0 norm of ∇u. [3] Namely, for $2-\epsilon<p<\infty$ and $u_0\in L^p(\mathbb{R}^n)$, $u=\mathcal{E}_A(u_0)$ is a global weak solution to (HC) with the equivalence

$$\|\nabla u\|_{T^p_0} \eqsim \|u_0\|_{L^p(\mathbb{R}^n)}. \tag{1.13}$$

[3]Their original solution class is given by $u\in X^p$ (the *Kenig–Pipher space*, as introduced in [KP93]). A complete theory of well-posedness and representation has been established within this solution class, even for $p=\infty$ and $u_0\in L^\infty(\mathbb{R}^n)$. Notably, the equivalence $\|u\|_{X^p}\eqsim\|u_0\|_{L^p(\mathbb{R}^n)}$ holds for $2-\epsilon<p\le\infty$.

Note that for $p = 2$, the tent space T^2_0 identifies with $L^2(\mathbb{R}^{1+n}_+)$, so this equivalence can be envisioned from the energy equality

$$\|u_0\|^2_{L^2(\mathbb{R}^n)} = 2\Re \int_0^\infty \int_{\mathbb{R}^n} (A(t,x)\nabla u(t,x)) \cdot \overline{\nabla u}(t,x) dt dx.$$

A later complement by Zatoń [Zat20] extended the equivalence (1.13) to $p = \infty$ with

$$\|\nabla u\|_{T^\infty_0} \eqsim \|u_0\|_{\mathrm{BMO}}.$$

Zatoń also established uniqueness and representation of weak solutions in the solution class $\nabla u \in T^p_0$. Namely, for $2-\epsilon < p \leq \infty$, any weak solution u to the equation $\partial_t u - \operatorname{div}(A\nabla u) = 0$ with $\nabla u \in T^p_0$ has a (unique) trace $u_0 \in L^p(\mathbb{R}^n)$ (or BMO if $p = \infty$), and u can be represented by the propagator solution map acting on u_0, *i.e.*, $u = \mathcal{E}_A(u_0)$ or equivalently,

$$u(t) = \Gamma_A(t,0)u_0, \quad t > 0.$$

Zatoń also extended these results to higher order parabolic systems.

A very recent work of Auscher and Portal [AP25] further considered source terms $F \in T^p_0$ for $2 \leq p \leq \infty$. For real coefficients, they extended this to $1 - \epsilon < p \leq \infty$.

Let us briefly outline the main ingredients in the works [AMP19, Zat20, AP25]. As noted in the final paragraph of Section 1.2.2, pointwise Gaussian decay fails for propagators associated with general complex coefficients. Instead, [AMP19] established localized L^2 Gaussian decay, also known as $L^2 - L^2$ *off-diagonal estimates* or *Davies–Gaffney estimates*, for the propagators $(\Gamma_A(t,s))$. More precisely, there exists a constant $c > 0$ so that for any $E, F \subset \mathbb{R}^n$ as Borel sets and $f \in L^2(\mathbb{R}^n)$,

$$\|\mathbb{1}_E \Gamma_A(t,s)\mathbb{1}_F f\|_{L^2(\mathbb{R}^n)} \lesssim \exp\left(-c\frac{\operatorname{dist}(E,F)^2}{t-s}\right) \|\mathbb{1}_F f\|_{L^2(\mathbb{R}^n)}.$$

These estimates were first introduced by Gaffney [Gaf59] and Davies [Dav92] for the semigroup generated by the Laplace–Betrami operator on Riemannian manifolds. In the autonomous case $A = A(x)$, the $L^2 - L^2$ off-diagonal estimates for the semigroup $(e^{-tL})_{t\geq 0}$ (generated by the elliptic operator $-L = \operatorname{div}(A(x)\nabla)$ as defined in (1.3)) were first exhibited in [AHL+02], playing a crucial role in the resolution of Kato's conjecture.

Combining such decay with square function estimates gives *a priori* estimates sufficient to establish the existence of weak solutions.

The proof of uniqueness and representation relies on an interior representation of weak solutions, which can be thought as an elaboration of Green's identity in the context of rough coefficients. More precisely, let u be a weak solution in a strip $(a,b) \times \mathbb{R}^n$ to the equation $\partial_t u - \operatorname{div}(A(t,x)\nabla u) = 0$ so that

$$\int_{\mathbb{R}^n} \left(\int_a^b \int_{B(x,b^{1/2})} |u(t,y)|^2 dt dy\right)^{1/2} e^{-\gamma|x|^2} dx < \infty \tag{1.14}$$

for some $\gamma \in (0, \frac{\Lambda_0^2}{16\Lambda_1^2(b-a)})$. Then u satisfies what we call as "*homotopy identity*"

$$u(t) = \Gamma_A(t,s)u(s) \ \text{ in } \mathscr{D}'(\mathbb{R}^n), \qquad \text{for } a < s < t < b,$$

in the sense that for any $h \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} u(t,x)\overline{h}(x)dx = \int_{\mathbb{R}^n} u(s,x)\overline{(\Gamma_A(t,s)^*h)}(x)dx. \tag{1.15}$$

Notice that the condition (1.14) resembles Aronson's condition (1.10) (for real coefficients), but it does not include a boundary condition. Therefore, the problem is reduced to understanding the boundary behavior of solutions and the continuity of propagators in compatible topologies.

1.2.5 Applications towards nonlinear PDE's

The model (1.1) plays an important role in the study of non-linear PDE's. For example, it helps to study existence and uniqueness of solutions to quasi-linear systems, using linearization and fixed-point arguments, see *e.g.*, [Lun95]. Some applications also touch free-boundary problems [PS16, DHMT21] and stochastic PDE's [AvNP14, PV19, AP25].

A particular interest stems from fluid mechanics, especially from the Navier–Stokes equations. For instance, Monniaux [Mon99] employed L^p-maximal regularity of the Laplacian to give an alternative proof of uniqueness of mild solutions to the 3D incompressible Navier–Stokes equation in $C^0_t L^3_x$, which was first obtained by Furioli, Lemarié-Rieusset and Terraneo [FLRT00].

Recently, Auscher and Frey [AF17] used the maximal regularity operator of the Laplacian on tent spaces to give a more operator-theoretic proof of Koch and Tataru's result on BMO^{-1} [KT01]. A very recent work of Danchin and Vasilyev [DV23] also use the maximal regularity operator on tent spaces to study the inhomogeneous incompressible Navier–Stokes equation.

Motivated by applications towards stochastic analysis, Portal and Veraar propose the following problem on well-posedness of the homogeneous Cauchy problem (HC) with initial data in homogeneous Hardy–Sobolev spaces $\dot{H}^{s,p}$, where s denotes regularity and p denotes integrability. Precise definition of these spaces is provided in Section 5.1. For context, in contrast with the usual identifications (modulo polynomials), we define the space $\dot{H}^{s,p}$ as a certain realization within *tempered distributions* $\mathscr{S}'(\mathbb{R}^n)$ of the Triebel–Lizorkin space $\dot{F}^s_{p,2}$. For $s = 0$, it is isomorphic to the Hardy space $H^p(\mathbb{R}^n)$ if $p \le 1$, the Lebesgue space $L^p(\mathbb{R}^n)$ if $1 < p < \infty$, and BMO if $p = \infty$. For $s < 0$, $\dot{H}^{s,\infty}$ is isomorphic to BMO^s introduced by Strichartz [Str80].

Problem 1.1 ([PV19, p.583]). Let $A \in L^\infty((0,\infty) \times \mathbb{R}^n; \mathrm{Mat}_n(\mathbb{C}))$ be uniformly elliptic. Does there exist a range of $s \ge 0$ and $p \in (1,\infty)$ so that the equivalence

$$\|u_0\|_{\dot{H}^{s,p}} \eqsim \|\nabla \mathcal{E}_A(u_0)\|_{T^p_{s/2}}$$

holds for any $u_0 \in \dot{H}^{s,p}$?

The equivalence clearly relates regularity of the initial data to the time weight in the tent space norm of the solution. In the special case $A = \mathbb{I}$, (*i.e.*, $\mathrm{div}(A\nabla)$ agrees with the Laplacian Δ), they intended to prove it for all $s \geq 0$ and $p \in (1, \infty)$, but the proof has a gap. In fact, as we shall see in Theorem 1.4, for $s \geq 1$ and $1 \leq p \leq \infty$, the equivalence $\|u_0\|_{\dot{H}^{s,p}} \eqsim \|\nabla e^{t\Delta} u_0\|_{T^p_{s/2}}$ holds if and only if u_0 is constant, for which $\|u_0\|_{\dot{H}^{s,p}} = 0$.

Another motivation for us to study this problem is that homogeneous Hardy–Sobolev spaces $\dot{H}^{s,p}$ naturally arise in non-linear PDE's as scale-invariant (or called *critical*) spaces. Addressing this problem could provide us with deep insights into a large class of non-linear PDE's. For instance, the incompressible Navier–Stokes equation admit a well-known hierarchy of critical spaces

$$\dot{H}^{\frac{n}{2}-1} \hookrightarrow L^n \hookrightarrow \dot{H}^{-1+\frac{n}{p},p} \hookrightarrow \mathrm{BMO}^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}, \quad 2 \leq n < p < \infty.$$

Here, the homogeneous Sobolev space $\dot{H}^{\frac{n}{2}-1}$ coincides with the homogeneous Hardy–Sobolev space $\dot{H}^{\frac{n}{2}-1,2}$ (up to constants), L^n coincides with $\dot{H}^{0,n}$, and BMO^{-1} coincides with $\dot{H}^{-1,\infty}$.

1.3 Main results

This section collects the main results of this thesis. We begin with the heat equation, followed by linear autonomous parabolic Cauchy problems and then the non-autonomous ones. Finally we present the results for the Navier–Stokes equation.

1.3.1 Heat equation

A fundamental topic in the study of the heat equation is to investigate the representation of heat solutions:

Problem 1.2. Given a heat solution u on the upper-half space $(0, \infty) \times \mathbb{R}^n$ or a strip $(0, T) \times \mathbb{R}^n$, when can we assert that u can be represented by the heat semigroup on the data, *i.e.*,

$$u(t) = e^{t\Delta} u_0 \tag{1.16}$$

for some u_0 and all $t \in (0, T)$?

The most general framework for such a representation is via tempered distributions. More precisely, given $u_0 \in \mathscr{S}'(\mathbb{R}^n)$, then $t \mapsto e^{t\Delta}u_0$ lies in $C^\infty([0,\infty);\mathscr{S}'(\mathbb{R}^n))$. Conversely, it has been shown in [Tay11, Chapter 3, Proposition 5.1] that any $u \in C^\infty([0,\infty);\mathscr{S}'(\mathbb{R}^n))$ solving the heat equation is represented by the heat semigroup applied to its initial value. Certainly, the argument still works in $C^1((0,\infty);\mathscr{S}'(\mathbb{R}^n)) \cap C([0,\infty);\mathscr{S}'(\mathbb{R}^n))$, which seems to close the topic. But it uses Fourier transform, so it is not transposable to more general equations (*e.g.*, parabolic equations with coefficients). Thus, one may wonder whether different concrete knowledge, like a growth condition, on the solution could lead to a representation, not using Fourier transform. Yet, one can observe that growth exceeding the inverse of a Gaussian when $|x| \to \infty$ is forbidden for the representation.

Another framework is that of non-negative solutions. A classical result by Widder [Wid44, Theorem 6] shows that in one-dimensional case, any non-negative C^2-solution u in the strip must be of the form (1.16) for some non-negative Borel measure u_0. It has been generalized to higher dimensions and classical solutions of parabolic equations with smooth coefficients by Krzyzanski [Krz64], via internal representation and a limiting argument. We are also going to use this idea below, but we want to remove the sign condition. Aronson later extended it to non-negative weak solutions of real parabolic equations, see [Aro68, Theorem 11].

Next, the uniqueness problem is tied with representation but they are different issues. For instance, let us mention the pioneering work on non-uniqueness by Tychonoff [Tyc35], and two works giving sufficient criteria on strips for uniqueness. One by Täcklind [Täc36] provides the optimal pointwise growth condition, and the other by Gushchin [Gus82] provides a local L^2 condition with prescribed growth, also optimal but more amenable to more general equations. In these results, the growth can be faster than the inverse of a Gaussian when $|x| \to \infty$, which hence excludes usage of tempered distributions, so uniqueness can hold without being able to represent general solutions.

With these observations in mind, it seems that we have two very different theories to approach representation (and uniqueness): one only using distributions and Fourier transform; one not using them at all.

Our first theorem (see Theorem 5.7) makes a bridge between them, *i.e.*, to obtain tempered distributions, not just measurable functions or measures, as initial data from local integrability conditions. Such conditions may only include integrability conditions in the interior, furnished by a uniform control.

Theorem 1.3 (Representation of heat solutions)**.** *Let* $0 < T \leq \infty$. *Let* $u \in \mathscr{D}'((0,T) \times \mathbb{R}^n)$ *be a distributional solution to the heat equation. Suppose that:*

(i) (Size condition) *For* $0 < a < b < T$*, there exist* $C(a,b) > 0$ *and* $0 < \gamma < 1/4$ *such that for any* $R > 0$*,*

$$\left(\int_a^b \int_{B(0,R)} |u(t,x)|^2 dtdx\right)^{1/2} \leq C(a,b) \exp\left(\frac{\gamma R^2}{b-a}\right);$$

(ii) (Uniform control) *There exists a sequence* (t_k) *tending to 0 such that* $(u(t_k))$ *is bounded in tempered distributions* $\mathscr{S}'$*.*

Then there exists a unique $u_0 \in \mathscr{S}'$ *so that* $u(t) = e^{t\Delta}u_0$ *for all* $0 < t < T$*.*

The key idea is to obtain an internal semigroup representation of caloric functions, *i.e.*, the homotopy identity (1.15) for $L = -\Delta$, from the L^2-growth on rectangles. Then the problem of representation of solutions is reduced to understanding their boundary behaviors approaching the initial time.

Applying this result to homogeneous Hardy–Sobolev spaces $\dot{H}^{s,p}$ provides us with a clear and strong correspondence between homogeneous Hardy–Sobolev spaces and tent spaces (see Theorem 5.17 and Corollary 5.19):

Theorem 1.4 (Heat equation and weighted tent spaces)**.** *Let* $s \in \mathbb{R}$*,* $0 < p \leq \infty$*, and* $\varphi \in \mathscr{S}'(\mathbb{R}^n)$*.*

(i) (Weighted tent-space estimates) *Suppose* $s < 1$*. If* $\varphi \in \dot{H}^{s,p}$*, then the function* $(t,x) \mapsto \nabla e^{t\Delta}\varphi(x)$ *belongs to* $T^p_{s/2}$ *with*

$$\|\nabla e^{t\Delta}\varphi\|_{T^p_{s/2}} \eqsim \|\varphi\|_{\dot{H}^{s,p}}.$$

(ii) (Representation of heat solutions) *Let* u *be a distributional solution to the heat equation on* $\mathbb{R}^{1+n}_+$ *with* $\nabla u \in T^p_{s/2}$*. Suppose* $s > -1$ *and* $\frac{n}{n+s+1} \leq p \leq \infty$*. Then there exists a unique* $u_0 \in \mathscr{S}'$ *so that* $u(t) = e^{t\Delta}u_0$ *for all* $t > 0$*. Moreover,*

(1) *If* $s \geq 1$ *and* $\frac{n}{n+s-1} \leq p \leq \infty$*, then* u *is a constant.*

(2) *If* $-1 < s < 1$ *and* $\frac{n}{n+s+1} \leq p \leq \infty$*, then there exist* $\varphi \in \dot{H}^{s,p}$ *and* $c \in \mathbb{C}$ *such that* $u_0 = \varphi + c$*, so* $u(t) = e^{t\Delta}\varphi + c$ *for all* $t > 0$*.*

Consequently, for $-1 < s < 1$ *and* $\frac{n}{n+s+1} \leq p \leq \infty$*, the map* $u_0 \mapsto (e^{t\Delta}u_0)(x) =: u(t,x)$ *is an isomorphism from* $\dot{H}^{s,p} + \mathbb{C}$ *onto the space of distributional solutions* u *to the heat equation with* $\nabla u \in T^p_{s/2}$*, and*

$$\|u_0\|_{\dot{H}^{s,p}/\mathbb{C}} \eqsim \|\nabla u\|_{T^p_{s/2}}.$$

The first point (i) shows a precise correspondence between the Sobolev regularity of initial data and the choice of the power weight. The second point (ii) shows the regularity range $s<1$ is essentially sharp.

For $s=0$, the first point (i) can be regarded as the characterization of homogeneous Hardy–Sobolev spaces via caloric extensions. It is reminiscent to the work of Fefferman and Stein [FS72] on defining Hardy spaces by conical square functions when translated to parabolic setting. But their point was to take extensions not related to any kind of equations to obtain an intrinsic definition of Hardy spaces. Here, we restrict ourselves to extensions related to the heat equation. It is also reminiscent to Littlewood–Paley theory using rather vertical square functions, see [Tri20, §4.1] for more on this topic.

Our method also applies to homogeneous Besov spaces $\dot{B}^s_{p,p}$ for s and p in the same range, by replacing tent spaces by Z-spaces (introduced by Barton and Mayboroda [BM16]) and using real interpolation. At the extreme point $s=1$ and $p=\infty$, it gives a new characterization of Lipschitz functions, see Proposition 5.24 and Theorem 5.25.

1.3.2 Autonomous parabolic Cauchy problems

Next, consider the autonomous parabolic Cauchy problems of divergence type

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(x)\nabla_x u) = f + \operatorname{div}_x F, & (t,x) \in (0,\infty)\times\mathbb{R}^n \\ u(0) = u_0 \end{cases} . \tag{1.17}$$

Assume that the coefficient matrix $A=A(x)\in L^\infty(\mathbb{R}^n;\mathrm{Mat}_n(\mathbb{C}))$ is uniformly elliptic, cf. (1.2). Denote by L the operator

$$L := -\operatorname{div}(A(x)\nabla)$$

on $L^2(\mathbb{R}^n)$ as defined in (1.3), and by $(e^{-tL})_{t\geq 0}$ the analytic semigroup generated by $-L$ on $L^2(\mathbb{R}^n)$.

1.3.2.1 Critical exponents

To precise our results, let us introduce some critical exponents. Recall that the L^p-theory of the semigroup (e^{-tL}) is ruled by four critical numbers, which are introduced in [Aus07, Proposition 3.15] for $1<p<\infty$, and later extended to $\frac{n}{n+1}<p<\infty$ in [AE23a, §6] to include Hardy spaces $H^p(\mathbb{R}^n)$. These numbers are

- $p_\pm(L)\in[\frac{n}{n+1},\infty]$ such that $(p_-(L),p_+(L))$ is the largest open set (an interval) of exponents p for which the semigroup $(e^{-tL})_{t\geq 0}$ is uniformly bounded on $L^p(\mathbb{R}^n)$ when $p>1$ and on $H^p(\mathbb{R}^n)$ when $p\leq 1$;

- $q_\pm(L) \in [\frac{n}{n+1}, \infty]$ such that $(q_-(L), q_+(L))$ is the largest open set (an interval) of exponents p for which the family $(t^{1/2}\nabla e^{-tL})_{t>0}$ is uniformly bounded on $L^p(\mathbb{R}^n)$ when $p > 1$ and on $H^p(\mathbb{R}^n)$ when $p \leq 1$.

It is known that $p_-(L) = q_-(L) < \frac{2n}{n+2}$, $q_+(L) > 2$, and $p_+(L) \geq \frac{nq_+(L)}{n-q_+(L)}$ [4]. The strict inequalities are best possible. Define

$$p_-^\flat(L) := \max\{p_-(L), 1\}.$$

The duality relation holds that $p_+(L^*) = p_-^\flat(L)'$, where p' denotes the Hölder conjugate of $p \in [1, \infty]$.

We extend the critical numbers to our $\dot{H}^{s,p}$-setting. For $-1 \leq s \leq 1$, define $p_\pm(s, L)$ as

$$\frac{1}{p_-(s,L)} := \begin{cases} \frac{1}{p_-(L)} + \frac{s}{n} & \text{if } 0 \leq s \leq 1 \\ \frac{1+s}{p_-(L)} - \frac{s}{q_+(L^*)'} & \text{if } -1 \leq s \leq 0 \end{cases},$$

and

$$p_+(s, L) := \max\{p_-(-s, L^*), 1\}'.$$

Notice that $p_\pm(0, L) = p_\pm(L)$, $p_-(-1, L) = q_+(L^*)' \in [1, 2)$, and $p_+(1, L) = q_+(L) \in (2, \infty]$.

We also introduce several other numbers which will parametrize our results. For convenience, we use a parameter β whose relation to the regularity exponent s is given by

$$\boxed{s = 2\beta + 1}$$

with $\beta > -1$ and no upper restriction. For $\beta > -1$, define the numbers $p_L(\beta) \in (0, 2)$ and $p_L^\flat(\beta) \in (0, 2)$ by

$$p_L(\beta) := \frac{np_-(L)}{n + (2\beta+1)p_-(L)}, \quad p_L^\flat(\beta) := \frac{np_-^\flat(L)}{n + (2\beta+1)p_-^\flat(L)},$$

which agree with each other when $p_-(L) \geq 1$. Write

$$\beta(L) := -\frac{1}{2} - \frac{n}{2}\left(\frac{1}{p_-(L)} - 1\right) \geq -1.$$

Note that $\beta(L) \geq -1/2$ if and only if $p_-(L) \geq 1$. Finally, we introduce the critical exponent $\widetilde{p}_L(\beta) \in (0, 2)$ so that:

(i) When $p_-(L) \geq 1$, it is given by

$$\widetilde{p}_L(\beta) := \begin{cases} p_L(\beta) & \text{if } \beta \geq -1/2 \\ p_-(2\beta+1, L) & \text{if } -1 < \beta < -1/2 \end{cases}.$$

[4] By convention, $p_+(L) = \infty$ if $q_+(L) \geq n$.

(ii) When $p_-(L) < 1$, it is given by

$$\widetilde{p}_L(\beta) := \begin{cases} p_L(\beta) & \text{if } \beta \geq \beta(L) \\ \frac{(\beta(L)+1)q_+(L^*)+\beta-\beta(L)}{(\beta(L)+1)q_+(L^*)} & \text{if } -1 < \beta < \beta(L) \end{cases}.$$

Remark that $\widetilde{p}_L(\beta(L)) = 1$ and $\widetilde{p}_L(-1) = q_+(L^*)' = p_-(-1, L)$.

In particular, for the negative Laplacian $L = -\Delta$, these numbers have explicit expressions. The critical numbers are

$$\begin{cases} p_-(-\Delta) = q_-(-\Delta) = \frac{n}{n+1}, \quad p_-^\flat(\Delta) = 1, \\ p_+(-\Delta) = q_+(-\Delta) = \infty, \end{cases}$$

so

$$p_-(s, -\Delta) = \frac{n}{n+s+1}, \quad p_+(s, -\Delta) = \infty, \quad -1 \leq s \leq 1.$$

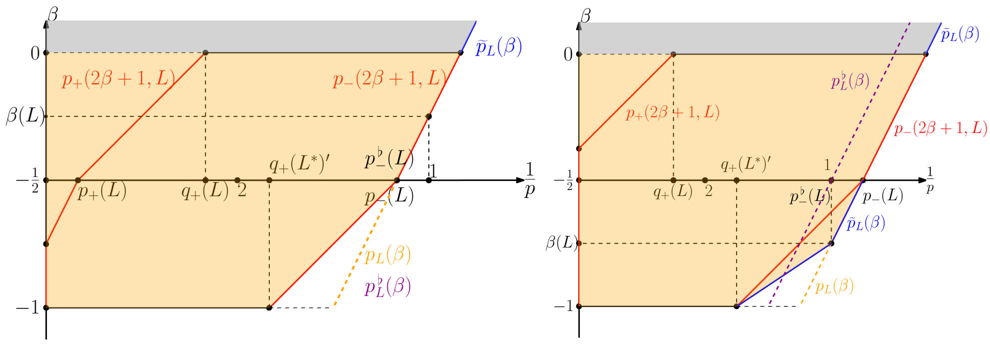
Hence, we have

$$\widetilde{p}_{-\Delta}(\beta) = p_{-\Delta}(\beta) = \frac{n}{n+2\beta+2}, \quad p_{-\Delta}^\flat(\beta) = \frac{n}{n+2\beta+1}, \quad \beta > -1.$$

Also, observe that for any L,

$$\widetilde{p}_L(\beta) \geq p_L(\beta) \geq \frac{n}{n+2\beta+2} = \widetilde{p}_{-\Delta}(\beta).$$

To illustrate these exponents, we give graphic representations in Figure 1.1, distinguishing the two cases $p_-(L) \geq 1$ and $p_-(L) < 1$. In these figures, we write p for $1/p$ to ease the presentation. When $p < 2$, we use red color for the graph of $p_-(2\beta+1, L)$, blue for that of $\widetilde{p}_L(\beta)$, and orange for that of $p_L(\beta)$ (and possibly purple for that of $p_L^\flat(\beta)$ when $p_-(L) < 1$). The orange shaded trapezoids are the regions of well-posedness for $\dot{H}^{2\beta+1,p}$-initial data, while the gray shaded is for constant initial data.



Case $p_-(L) \geq 1$ Case $p_-(L) < 1$

Figure 1.1: Critical exponents for autonomous parabolic Cauchy problems

Interestingly, the (smaller) set delimited by the red lines (excluded) and the black lines (included) has a special signification and is called the *identification range* for the operator L, see Section 6.4 and in particular Proposition 6.36. [5]

[5] In the case where $p > 2$, the red colored broken lines for the values of $p_+(2\beta+1, L)$

1.3.2.2 Main results

The following theorem presents well-posedness of autonomous parabolic Cauchy problems of type (1.17), see Theorems 6.11, 6.22, and 6.33 for more details.

Theorem 1.5 (Well-posedness of autonomous parabolic Cauchy problems)**.** *Let $\beta > -1$ and $\widetilde{p}_L(\beta) < p \le \infty$. Let $\gamma > -1/2$ and $p^\flat_L(\gamma) < q \le \infty$. Suppose*

$$\gamma \ge \beta, \quad 2\beta - \frac{n}{p} = 2\gamma - \frac{n}{q}.$$

(i) *If $\beta < 0$, then for any $u_0 \in \dot{H}^{2\beta+1,p}$, $F \in T^p_{\beta+1/2}$, and $f \in T^q_\gamma$, there exists a unique global weak solution u to the Cauchy problem*

$$\begin{cases} \partial_t u - \operatorname{div}(A(x)\nabla u) = f + \operatorname{div} F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases},$$

so that $\nabla u \in T^p_{\beta+1/2}$. Moreover, the estimate holds that

$$\|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|u_0\|_{\dot{H}^{2\beta+1,p}} + \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^q_\gamma},$$

and u lies in $C([0,\infty); \mathscr{S}')$. If $u_0 = 0$, then u also lies in $T^p_{\beta+1}$ with

$$\|u\|_{T^p_{\beta+1}} \lesssim \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^q_\gamma}.$$

(ii) *If $\beta \ge 0$, then the same statement holds when u_0 is constant (for which $\|u_0\|_{\dot{H}^{2\beta+1,p}} = 0$).*

In fact, when $\beta \ge 0$, as we shall see in Theorem 1.7 (i), constant initial data are the only ones compatible with the solution class $\nabla u \in T^p_{\beta+1/2}$.

As $t \to 0$, the convergence of $u(t)$ to u_0 can be refined to finer topology than $\mathscr{S}'$, depending on β and p. For instance, consider homogeneous Cauchy problems ($f = 0$, $F = 0$). Our weak solutions are constructed by extension of the *semigroup solution map* $\mathcal{E}_L$, which is initially defined from $L^2(\mathbb{R}^n)$ to $L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}}(\mathbb{R}^n))$ by

$$\mathcal{E}_L(u_0)(t,x) := (e^{-tL}u_0)(x), \quad t > 0,\ x \in \mathbb{R}^n. \tag{1.18}$$

When $(2\beta+1, p)$ lies in the identification range, it preserves $\dot{H}^{2\beta+1,p}$-regularity, *i.e.*, when $u_0 \in \dot{H}^{2\beta+1,p}$, [6]

$$\mathcal{E}_L(u_0) \in C_0([0,\infty); \dot{H}^{2\beta+1,p}) \cap C^\infty((0,\infty); \dot{H}^{2\beta+1,p}).$$

---

are obtained by symmetry about the point $(1/p, \beta) = (1/2, -1/2)$ from the ones for $\max\{p_-(-2\beta-1, L^*), 1\}$. Thus it depends on the value of $p_-(L^*)$. In the first (resp. second) picture, this corresponds to having $p_-(L^*) \ge 1$ (resp. $p_-(L^*) < 1$). As the values of $p_-(L)$ and $p_-(L^*)$ are independent, we should have in fact 4 such figures.

[6]Here, $C_0([0,\infty); E)$ is the space of continuous functions with limit 0 as $t \to \infty$ in the prescribed topology of E.

In this range with $p \geq 1$ and $2\beta < n/p$, [7] it also basically says $v = \mathcal{E}_L(u_0)$ is a strong solution of the abstract Cauchy problem

$$\begin{cases} \partial_t v + L_{\beta,p} v = 0 & \text{in } \dot{H}^{2\beta+1,p} \\ v(0) = v_0 \end{cases},$$

where $-L_{\beta,p}$ is the generator of the extended semigroup on $\dot{H}^{2\beta+1,p}$. Note that $p < 1$ is possible here, even though there is no abstract theory for strong solutions in quasi-Banach spaces as far as we know. But when $(2\beta+1,p)$ does not lie in the identification range, the extension $\mathcal{E}_L$ may not be interpreted as a semigroup.

For inhomogeneous Cauchy problems ($u_0 = 0$, $F = 0$), our weak solutions are constructed by the extension of the Duhamel operator $\mathcal{L}_1$ defined in (1.5) to the tent space T^q_γ. In fact, when $\gamma > -1/2$ and $p_L^\flat(\gamma) < p \leq \infty$, for any $f \in T^q_\gamma$, the weak solution $u = \mathcal{L}_1(f)$ satisfies the estimate

$$\|u\|_{T^q_{\gamma+1}} + \|\nabla u\|_{T^q_{\gamma+1/2}} \lesssim \|f\|_{T^q_\gamma}.$$

Moreover, both $\partial_t u$ and $\operatorname{div}(A\nabla u)$ (in the sense of distributions) belong to T^q_γ with

$$\|\partial_t u\|_{T^q_\gamma} + \|\operatorname{div}(A\nabla u)\|_{T^q_\gamma} \lesssim \|f\|_{T^q_\gamma}. \tag{1.19}$$

We call (1.19) as the maximal regularity estimate on tent spaces. Indeed, since the exponent γ is related to regularity, this estimate indicates that $\partial_t u$ and $\operatorname{div}(A\nabla u)$ can not simultaneously lie in a space of higher regularity, otherwise the regularity of f could be raised. It is exactly the spirit of maximal regularity, although the interpretation is no longer via semigroup theory as in the L^p-maximal regularity theory.

To obtain these estimates, we develop a theory of singular integral operators on tent spaces. The prototypical example illustrating the theory is the maximal regularity operator $\mathcal{L}_0$ defined in (1.7). It was De Simon [dS64] who first proved that $\mathcal{L}_0$ extends to a bounded operator on $L^2((0,\infty); L^2(\mathbb{R}^n)) \simeq T^2_0$. Auscher, Kriegler, Monniaux and Portal [AKMP12] further investigated bounded extension of $\mathcal{L}_0$ on tent spaces T^p_β for a certain range of β and p with explicit bounds.

Our framework refines the definition of singular integral operators by exploiting the use of $L^p - L^q$ off-diagonal decay, which rectifies an inaccuracy present in the framework of [AKMP12]. We also develop a new approach to prove bounded extension. The main strategy is to establish a pointwise estimate that bounds the conical square function in the tent space norm by the *vertical square function* (also known as the *Littlewood–Paley–Stein functional*), to which we subsequently apply Rubio de Francia's extrapolation

[7] When $2\beta - n/p \geq 0$, our space $\dot{H}^{2\beta+1,p}$ is not a Banach space. It is semi-normed.

and Stein's interpolation. This approach improves the previous results in [AMP12, AKMP12]; for instance, we give an enhanced (possibly optimal) result (see Proposition 6.17) of boundedness of the maximal regularity operator on tent spaces:

Theorem 1.6 (Boundedness of the maximal regularity operator on tent spaces)**.** *Let $\gamma > -1/2$ and $p_L^\flat(\gamma) < p \leq \infty$. Then $\mathcal{L}_0$ extends to a bounded operator on T^p_γ.*

For $\gamma = 0$, we recover the results by Huang [Hua17] with a simpler proof. We also remark that the condition $\beta > -1/2$ is optimal, see [AA11b] in the case $p = 2$ (where T^2_β coincides with $L^2_\beta(\mathbb{R}^{1+n}_+)$).

We also establish the representation (hence uniqueness) of weak solutions in this class. For convenience, we set $\mathcal{E}_L(c) = c$ for any constant function c.

Theorem 1.7 (Representation of weak solutions to autonomous parabolic equations)**.** *Let $\beta > -1$ and $\widetilde{p}_L(\beta) < p \leq \infty$. Let u be a weak solution to the equation*

$$\partial_t u - \operatorname{div}_x(A(x)\nabla_x u) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n,$$

with $\nabla u \in T^p_{\beta+1/2}$. Then u has a trace $u_0 \in \mathscr{S}'$ in the sense that $u(t)$ converges to u_0 in $\mathscr{S}'$ as $t \to 0$. Moreover,

(i) *If $\beta \geq 0$ and $\frac{n}{n+2\beta} \leq p \leq \infty$, then u is a constant.*

(ii) *If $-1 < \beta < 0$, then there exist $\varphi \in \dot{H}^{2\beta+1,p}$ and $c \in \mathbb{C}$ so that $u_0 = \varphi + c$ and $u = \mathcal{E}_L(\varphi) + c$, where $\mathcal{E}_L$ is an appropriate extension of the semigroup solution map given by (1.18).*

Consequently, for $-1 < \beta < 0$ and $\widetilde{p}_L(\beta) < p \leq \infty$, the map $u_0 \mapsto \mathcal{E}_L(u_0) = u$ is an isomorphism from $\dot{H}^{2\beta+1,p} + \mathbb{C}$ onto the space of global weak solutions to $\partial_t u - \operatorname{div}(A\nabla u) = 0$ with $\nabla u \in T^p_{\beta+1/2}$, and

$$\|u_0\|_{\dot{H}^{2\beta+1,p}/\mathbb{C}} \eqsim \|\nabla u\|_{T^p_{\beta+1/2}}.$$

For the endpoint case $\beta = -1$, we also prove existence of weak solutions u to the homogeneous Cauchy problem ($f = 0$, $F = 0$) with initial data $u_0 \in \dot{H}^{-1,p}$ for $p_-(-1, L) = q_+(L^*)' < p \leq \infty$, see Proposition 6.52. But the problem of uniqueness and representation remains widely open.

Analogous results for initial data in homogeneous Besov spaces $\dot{B}^s_{p,p}$ are also exhibited with the same range of s and p.

1.3.3 Non-autonomous parabolic Cauchy problems

Consider the non-autonomous parabolic Cauchy problems of type (1.1)

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(t,x)\nabla_x u) = f + \operatorname{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases},$$

where $A \in L^\infty((0,\infty) \times \mathbb{R}^n; \mathrm{Mat}_n(\mathbb{C}))$ is uniformly elliptic.

Let us first introduce the critical exponents. Denote by $(\Gamma_A(t,s))_{0 \leq s \leq t < \infty}$ the propagators associated to A defined in (1.11). Let $p_\pm(A)$ be two real numbers in $[1,\infty]$ so that $(p_-(A), p_+(A))$ is the largest open set of exponents p for which $(\Gamma_A(t,s))$ extends to a uniformly bounded family on $L^p(\mathbb{R}^n)$. It is known that $p_-(A) < 2$ and $p_+(A) > 2$. For time-independent coefficients, they correspond to the critical numbers $p_-^\flat(L)$ and $p_+(L)$, respectively.

We also parametrize our results by $\beta \in \mathbb{R}$, whose relation with the regularity index s is given by

$$\boxed{s = 2\beta + 1}.$$

Define the number

$$p_A(\beta) := \frac{np_-(A)}{n + (2\beta+1)p_-(A)}.$$

Let $\zeta < -1/2$ be a fixed reference number. Define the critical exponents $p_\zeta^\pm(\beta)$ by

$$p_\zeta^-(\beta) := \begin{cases} \frac{2(2\zeta+1)p_-(A)}{4(\zeta-\beta)+(2\beta+1)p_-(A)} & \text{if } \zeta \leq \beta < -1/2 \\ p_A(\beta) & \text{if } \beta \geq -1/2 \end{cases},$$

and

$$p_\zeta^+(\beta) := \begin{cases} \frac{2(2\zeta+1)}{2\beta+1} & \text{if } \zeta \leq \beta < -1/2 \\ \infty & \text{if } \beta \geq -1/2 \end{cases}.$$

Note that $p_\zeta^-(\zeta) = p_\zeta^+(\zeta) = 2$. To illustrate these exponents, we give graphic representations in Figure 1.2. In this figure, we also write p for $1/p$ to ease the presentation. We use red dashed line for the graph of $p_\zeta^-(\beta)$ for $\beta < -1/2$, red normal line for that of $p_\zeta^+(\beta)$, and blue dashed line for that of $p_A(\beta)$. Parallel lines to the blue dashed line are lines of embedding for Hardy–Sobolev spaces and weighted tent spaces going downward.

We shall introduce a new parameter $\beta_A \in [-1, -1/2)$, which only depends on the ellipticity of A and the dimension n, as the lower bound of β. Taking $\zeta = \beta_A$, the orange shaded trapezoid becomes the region of well-posedness for $\dot{H}^{2\beta+1,p}$-initial data, while the blue shaded is for constant initial data.

In particular, consider the orange triangle below the line $\beta = -1/2$. We have stated things in a way that it is above two segments. Actually, this is an artifact of our statements made to simplify the exposition. One could state things in a way that this orange triangle is above some convex curve that passes through $(0, -1/2)$, $(1/2, \beta_A)$, and $(1/p_-(A), -1/2)$. We do not know what this precise curve should look like. In fact, for $\beta < -1/2$, the red normal line for $p_{\beta_A}^+(\beta)$ is only due to our choice of β_A. If instead we take the lower bound and if it is not attained, then this line should be dashed, but the point $(0, -1/2)$ is always included.

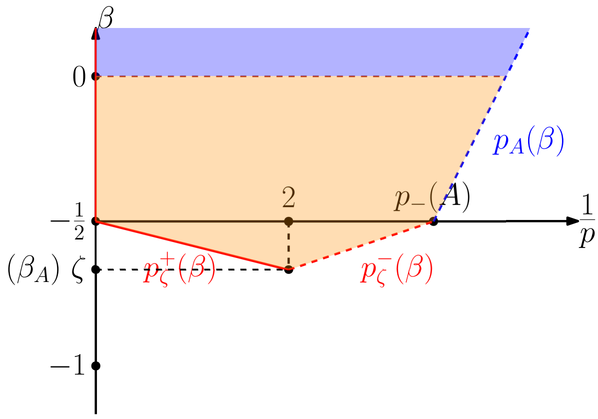


Figure 1.2: Critical exponents for non-autonomous parabolic Cauchy problems

This limitation is due to the absence of exponents expressing boundedness of $\nabla\Gamma_A(t,s)$ for fixed s,t, which is a substantial difference with the autonomous case. This makes the analysis much more complicated.

Let us now present well-posedness for non-autonomous Cauchy problems of type (1.1), as announced.

Theorem 1.8 (Well-posedness of non-autonomous parabolic Cauchy problems)**.** *There exists $\beta_A \in [-1,-1/2)$ only depending on the ellipticity of A and the dimension n so that the following properties hold. Let $\beta > \beta_A$ and $p^-_{\beta_A}(\beta) < p \le p^+_{\beta_A}(\beta)$. Let $\gamma > -1/2$ and $p_A(\gamma) < q \le \infty$. Suppose*

$$\gamma \ge \beta, \quad 2\gamma - \frac{n}{q} = 2\beta - \frac{n}{p}.$$

(i) *If $\beta < 0$, then for any $u_0 \in \dot{H}^{2\beta+1,p}$, $F \in T^p_{\beta+1/2}$, and $f \in T^q_\gamma$, there exists a unique global weak solution u to the Cauchy problem*

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(t,x)\nabla_x u) = f + \operatorname{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases}$$

so that $\nabla u \in T^p_{\beta+1/2}$. Moreover, the estimate holds that

$$\|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|u_0\|_{\dot{H}^{2\beta+1,p}} + \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^q_\gamma},$$

and u lies in $C([0,\infty);\mathscr{S}')$. If $u_0 = 0$, then u also lies in $T^p_{\beta+1}$ with

$$\|u\|_{T^p_{\beta+1}} \lesssim \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^q_\gamma}.$$

(ii) *If $\beta \ge 0$, then the above statements also holds when u_0 is constant (for which $\|u_0\|_{\dot{H}^{2\beta+1,p}} = 0$).*

For $\beta \geq 0$, the only initial data compatible with the solution class $\nabla u \in T^p_{\beta+1/2}$ are constant, see Theorem 1.9 (i).

For $\beta = -1/2$ ($s = 0$), we recover the results in [AMP19, Zat20] with a conceptually simpler and more operator-theoretic proof. It also furnishes the results in [AP25] for Lions' equation ($u_0 = 0$, $f = 0$) with the range $p_-(A) < p < 2$.

For inhomogeneous Cauchy problems ($u_0 = 0$, $F = 0$), our weak solutions are also constructed by the extension of the Duhamel operator $\mathcal{L}^A_1$, which is initially defined from $L^2(\mathbb{R}^{1+n}_+)$ to $L^\infty_{\mathsf{loc}}([0,\infty); L^2(\mathbb{R}^n))$ by the Bochner integrals

$$\mathcal{L}^A_1(f)(t) := \int_0^t \Gamma_A(t,s) f(s) ds, \quad t > 0,$$

to the tent space T^q_γ. We also show that when $\gamma > -1/2$ and $p_A(\gamma) < q \leq \infty$, for any $f \in T^q_\gamma$, the weak solution $u = \mathcal{L}^A_1(f)$ satisfies the estimate

$$\|u\|_{T^q_{\gamma+1}} + \|\nabla u\|_{T^q_{\gamma+1/2}} \lesssim \|f\|_{T^q_\gamma}.$$

The proof is also based on the theory of singular integral operators on tent spaces. But at this stage, we do not have maximal regularity estimates for tent space norms of $\partial_t u$ and $\operatorname{div}(A\nabla u)$ as in (1.6), since we do not have appropriate estimates for the operator $\varphi \mapsto \operatorname{div}(A\nabla\Gamma_A(t,s)\varphi)$, due to the lack of analyticity of the propagators.

The next theorem exhibits representation of weak solutions.

Theorem 1.9 (Representation of weak solutions to non-autonomous parabolic Cauchy problems)**.** *Let β_A be the constant given in Theorem 1.8. Let $\beta > \beta_A$ and $p^-_{\beta_A}(\beta) < p \leq p^+_{\beta_A}(\beta)$. Let u be a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = 0$ with $\nabla u \in T^p_{\beta+1/2}$. Then u has a trace $u_0 \in \mathscr{S}'$, in the sense that $u(t)$ converges to u_0 in $\mathscr{S}'$ as $t \to 0$. Moreover,*

(i) *If $\beta \geq 0$ and $\frac{n}{n+2\beta} \leq p \leq \infty$, then u is a constant.*

(ii) *If $\beta_A < \beta < 0$, then there exist $\varphi \in \dot{H}^{2\beta+1,p}$ and $c \in \mathbb{C}$ so that $u_0 = \varphi + c$ and $u = \mathcal{E}_A(\varphi) + c$, where $\mathcal{E}_A$ is the extension of the propagator solution map defined by (1.12).*

Consequently, for $\beta_A < \beta < 0$ and $p^-_{\beta_A}(\beta) < p \leq p^+_{\beta_A}(\beta)$, the map $u_0 \mapsto \mathcal{E}_A(u_0) = u$ is an isomorphism from $\dot{H}^{2\beta+1,p} + \mathbb{C}$ onto the space of global weak solutions to $\partial_t u - \operatorname{div}(A\nabla u) = 0$ with $\nabla u \in T^p_{\beta+1/2}$, and

$$\|u_0\|_{\dot{H}^{2\beta+1,p}/\mathbb{C}} \eqsim \|\nabla u\|_{T^p_{\beta+1/2}}.$$

This finally solves Problem 1.1, even with extra ranges for $s < 0$ ($\beta < -1/2$) and $p \leq 1$. Let us also mention an interesting corollary that we did not find in the literature. Note that for $p = 2$, the tent space T^2_β identifies with the time-weighted L^2-space

$$T^2_\beta \simeq L^2\left((0,\infty), t^{-2\beta}dt;\ L^2(\mathbb{R}^n)\right) =: L^2_\beta(\mathbb{R}^{1+n}_+), \quad \beta \in \mathbb{R}.$$

In this special case, Theorem 1.8 yields

Corollary 1.10. *Let $\beta > -1/2$. For any $f \in L^2_\beta(\mathbb{R}^{1+n}_+)$ and $F \in L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$, there exists a unique global weak solution u to the Cauchy problem*

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(t,x)\nabla_x u) = f + \operatorname{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = 0 \end{cases}$$

so that $\nabla u \in L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$. Moreover, u belongs to $L^2_{\beta+1}(\mathbb{R}^{1+n}_+)$ with

$$\|u\|_{L^2_{\beta+1}(\mathbb{R}^{1+n}_+)} + \|\nabla u\|_{L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)} \lesssim \|F\|_{L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)} + \|f\|_{L^2_\beta(\mathbb{R}^{1+n}_+)}.$$

The same results are also valid for $\beta_A < \beta \leq -1/2$ and $f = 0$.

Let us briefly compare our results with the series of works by Krylov *et al.* [Kry07a, Kry07b, KK07, Kry08, DK11a, DK11b, DK18]. They construct the generalized weak solutions (with null initial data) by studying the resolvent of the parabolic operator $\partial_t - \operatorname{div}(A\nabla)$, *i.e.*, the inverse of $\lambda + \partial_t - \operatorname{div}(A\nabla)$ for a range of $\lambda \in [0,\infty)$, on $L^p_t L^q_x$. Their approach covers all $p, q \in (1,\infty)$ under the VMO condition, and extends to elliptic/parabolic equations of non-divergence type, etc.

By contrast, we exploit off-diagonal decay of propagators within a limited range of p, thereby relaxing the VMO condition. We further employ (conical) square function estimates from tent space theory to derive finer estimates, notably the relation between the weight in time in the tent space norm of the solution and the regularity of the initial data. This reveals the homogeneity inherent in both the equation and the initial data. However, our method is confined to equations of divergence type. We do not treat lower-order terms (*e.g.*, $(b \cdot \nabla)u + cu$, which break the homogeneity of the equation) or bounded domains (which affect the homogeneity of the initial data). Extending the framework to such inhomogeneous settings remains to be explored.

1.3.4 Navier–Stokes equations

Consider the Cauchy problem of the incompressible Navier–Stokes equation

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = \Delta u - \nabla p, \\ \nabla \cdot u = 0, \\ u(0) = u_0, \end{cases} \tag{NS}$$

where $u : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the unknown velocity vector, and $p : (0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ denotes the unknown scalar pressure.

Initiated from Fujita and Kato [FK64, Kat84], there have been numerous works on well-posedness or ill-posedness of the Navier-Stokes equation in scale-invariant (or called *critical*) spaces, see [Che92, KP94, Can97, Pla98, BP08, Yon10, Wan15]. Up to now, the optimal small-data global existence result was obtained by Koch and Tataru [KT01] in the space BMO^{-1}, which consists of divergence of BMO-vector fields. More precisely, they proved that for small initial data $u_0 \in \mathrm{BMO}^{-1}$, there is a global mild solution u to (NS) with initial data u_0 so that

$$\|u\|_{X_\infty} := \sup_{t>0} \|t^{1/2} u(t)\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{T^\infty_0} < \infty.$$

Since then, there has been a strong interest in understanding the regularity of Koch–Tataru solutions. Miura and Sawada [MS06], and Germain, Pavlović and Staffilani [GPS07] obtained spatial analyticity of u. Despite the huge success, there remains a long-standing problem on time regularity of u. The best result was obtained by Auscher, Dubois and Tchamitchian [ADT04], which asserts that *any* mild solution $u \in X_\infty$ with initial data $u_0 \in \mathrm{BMO}^{-1}$ belongs to $L^\infty((0,\infty); \mathrm{BMO}^{-1})$.

The following theorem solves this problem and demonstrates at the same time its long-time behavior (see Theorems 8.1 and 8.2).

Theorem 1.11. *Let $u_0 \in \mathrm{BMO}^{-1}$ be divergence free and $u \in X_\infty$ be a mild solution to the Navier–Stokes equation with initial data u_0. Then u belongs to $C_{\mathsf{w}}([0,\infty); \mathrm{BMO}^{-1})$ [8] with $u(0) = u_0$, and $u(t)$ tends to 0 in BMO^{-1} as $t \to \infty$. Here, BMO^{-1} is endowed with the weak*-topology with respect to the homogeneous Hardy–Sobolev space $\dot{H}^{1,1}$.*

In fact, the argument also applies to any strip $(0,T) \times \mathbb{R}^n$ for $0 < T \leq \infty$.

This result is sharp. For continuity, notice that the heat semigroup $(e^{t\Delta})$ is only weak*-continuous on BMO^{-1}, so the weak*-continuity of $u(t)$ in BMO^{-1} might possibly be the best expected result. Regarding long-time behavior, self-similar solutions with small initial data in BMO^{-1} provide counterexamples demonstrating that the decay fails in the strong topology of BMO^{-1}.

We emphasize that no smallness assumptions are imposed on the initial data nor the solution itself. The proof is purely based on duality between the tent spaces T^∞_0 and T^1_0. Main ingredients include the correspondence between homogeneous Hardy–Sobolev spaces and tent spaces (cf. Theorem 1.4) and a new decomposition of the Oseen kernel introduced by Auscher and Frey [AF17], using the maximal regularity operator of the Laplacian on tent spaces.

[8] Here, $C_{\mathsf{w}}([0,\infty); \mathrm{BMO}^{-1})$ is the space of weakly* continuous functions valued in BMO^{-1}.

1.4 Organization, citations, and notations

1.4.1 Organization

The organization of this thesis is as follows.

Chapter 3 briefly reviews and refines some fundamental properties of tent spaces to be frequently used.

The first part (Chapter 4) is devoted to developing the theory of singular integral operators on tent spaces. Main results are collected in Section 4.2.1.

In the second part (Chapter 5, 6, and 7), we investigate the linear parabolic Cauchy problems of type (1.1). Chapter 5 is concerned with the heat equation.

Chapter 6 considers the autonomous case. By linearity, we split (1.1) into three models: inhomogeneous Cauchy problems ($u_0 = 0$, $F = 0$), the Lions equation ($u_0 = 0$, $f = 0$), and homogeneous Cauchy problems ($f = 0$, $F = 0$). Main properties of weak solutions to these three problems are presented in Theorem 6.11, Theorem 6.22, and Theorem 6.33, respectively. Section 6.6 contains extension to initial data in homogeneous Besov spaces, and Section 6.7 discusses the homogeneous Cauchy problem at the endpoint for regularity index $\beta = -1$.

Chapter 7 deals with the non-autonomous counterparts. Main results of weak solutions are exhibited in Theorem 7.11 for inhomogeneous Cauchy problems, Theorem 7.15 for the Lions equation, and Theorem 7.26 for homogeneous Cauchy problems.

The final part (Chapter 8) discusses applications to the incompressible Navier–Stokes equation.

1.4.2 Citations

This thesis is a compilation of the following publications and preprints:

1. [AH24] *On representation of solutions to the heat equation*, with Pascal Auscher, **Comptes Rendus Mathématique**, Vol.362 (2024), pp.761-768. DOI: 10.5802/crmath.593. arXiv: 2310.19330.

2. [AH25a] *On well-posedness and maximal regularity for parabolic Cauchy problems on weighted tent spaces*, with Pascal Auscher, **Journal of Evolution Equations**, 25, 16 (2025). DOI: 10.1007/s00028-024-01041-x. arXiv: 2311.04844.

3. [AH25b] *On well-posedness for parabolic Cauchy problems of Lions type with rough initial data*, with Pascal Auscher, **Mathematische Annalen** (2025). DOI: 10.1007/s00208-025-03149-y. arXiv: 2406.15775.

4. [Hou24] *On regularity of solutions to the Navier–Stokes equation with initial data in* BMO^{-1}. arXiv: 2410.16468.

5. [Hou25] *On well-posedness for non-autonomous parabolic Cauchy problems with rough initial data.* arXiv: 2505.09387.

1.4.3 Notations

Throughout this thesis, for any $p, q \in (0, \infty]$, we write

$$[p, q] := \frac{1}{p} - \frac{1}{q},$$

if there is no confusion with closed intervals.

We say $X \lesssim Y$ (or $X \lesssim_A Y$, resp.) if $X \leq CY$ with an irrelevant constant C (or depending on A, resp.), and say $X \eqsim Y$ if $X \lesssim Y$ and $Y \lesssim X$.

For any (Euclidean) ball $B \subset \mathbb{R}^n$, write $r(B)$ for the radius of B.

Let (X, μ) be a measure space. For any measurable subset $E \subset X$ with finite positive measure and $f \in L^1(E, \mu)$, we write

$$\fint_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu.$$

Write

$$\| \cdot \|_p := \| \cdot \|_{L^p(X,\mu)}.$$

Write $\mathbb{R}^{1+n}_+ := \mathbb{R}_+ \times \mathbb{R}^n = (0, \infty) \times \mathbb{R}^n$. For any function f defined on $\mathbb{R}^{1+n}_+$, denote by $f(t)$ the function $x \mapsto f(t, x)$ for any $t > 0$. For any $\beta \in \mathbb{R}$ and $E \subset \mathbb{R}^{1+n}_+$, denote by

$$L^p_\beta(E) := L^p(E, t^{-p\beta} dtdy).$$

Often, we omit the unweighted Lebesgue measure in the integral and the domain in the function space, if it is clear from the context. We use the sans-serif font c in the scripts of function spaces in short of "with compact support" in the prescribed set, and loc if the prescribed property holds on all compact subsets of the prescribed set.

Chapter 2

Introduction (en Français)

"*Parler les langues étrangères, voyez-vous, c'est vouloir marcher lorsqu'on est boiteux.*"

Sechs Vorträge aus der reinen Mathematik und mathematischen Physik,
Henri Poincaré

2.1 Objectifs principaux

L'objectif principal de cette thèse est d'étudier la continuité des opérateurs d'intégrale singulière sur les espaces de tentes et d'explorer leurs applications aux équations aux dérivées partielles (EDP) paraboliques, incluant les problèmes de Cauchy paraboliques linéaires sous forme divergence et l'équation de Navier–Stokes incompressible non-linéaire.

La première partie de cette thèse est consacrée à l'établissement d'une théorie des opérateurs d'intégrale singulière sur les espaces de tentes reposant sur la notion de décroissance hors diagonale $L^p - L^q$. Nous étendons et améliorons les travaux pionniers d'Auscher et al. [AKMP12]; par exemple, notre théorie s'intéresse l'étude des EDP (en particulier paraboliques), tels que l'opérateur de Duhamel et son gradient ainsi que l'opérateur de régularité maximale. En particulier, nous obtenons un résultat de bornitude presque optimal pour ces opérateurs sur les espaces de tentes.

Dans la deuxième partie de cette thèse, nous établissons un cadre théorique pour l'étude des problèmes de Cauchy paraboliques sous forme divergence

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(t,x)\nabla_x u) = f + \operatorname{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases}, \tag{2.1}$$

en utilisant la théorie des opérateurs d'intégrale singulière sur les espaces de tentes. Nous proposons un panorama complet de l'existence, de l'unicité et de la représentation des solutions faibles (d'énergie) de (2.1). La matrice de coefficients à valeurs complexes $A(t,x)$ est supposée uniformément elliptique, mais seulement bornée et mesurable tant en espace et en temps. Plus précisément, on appelle $A \in L^\infty((0,\infty) \times \mathbb{R}^n; \mathrm{Mat}_n(\mathbb{C}))$ *uniformément elliptique* s'il existe $\Lambda_0, \Lambda_1 > 0$ tels que pour presque tout $(t,x) \in (0,\infty) \times \mathbb{R}^n$ et tous $\xi, \eta \in \mathbb{C}^n$,

$$\Re(\langle A(t,x)\xi, \xi\rangle) \geq \Lambda_0 |\xi|^2, \quad |\langle A(t,x)\xi, \eta\rangle| \leq \Lambda_1 |\xi||\eta|. \tag{2.2}$$

Il est à noter que nous n'imposons aucune hypothèse de symétrie ni de régularité sur les coefficients, ni par rapport à t, ni par rapport à x.

Au titre de la condition initiale $u(0) = u_0$, on suppose que $u(t)$ tende vers 0 lorsque $t \to 0$ au sens des distributions, c'est-à-dire dans $\mathscr{D}'(\mathbb{R}^n)$.

Ce modèle sert d'exemple simplifié mais représentatif pour les systèmes paraboliques. En fait, nos résultats s'étendent naturellement aux systèmes paraboliques pour lesquels la condition d'ellipticité est remplacée par une inégalité de Gårding sur $\mathbb{R}^n$, uniforme par rapport à t: il existe $\Lambda_0 > 0$ tel que pour presque tout $t > 0$ et toute $\varphi \in \dot{H}^1(\mathbb{R}^n; \mathbb{C}^m)$,

$$\Re \int_{\mathbb{R}^n} (A(t,x)\nabla\varphi(x)) \cdot \overline{\nabla\varphi}(x) \geq \Lambda_0 \|\nabla\varphi\|^2_{L^2(\mathbb{R}^n)}.$$

Ici,

$$A(t,x) = (A^{\alpha,\beta}_{i,j}(t,x))^{1\leq\alpha,\beta\leq n}_{1\leq i,j\leq m} \in L^\infty((0,\infty)\times\mathbb{R}^n; \mathrm{Mat}_{mn}(\mathbb{C}))$$

et l'on utilise la notation

$$(A(t,x)\nabla\varphi(x)) \cdot \overline{\nabla\varphi}(x) := \sum_{\substack{1\leq i,j\leq m \\ 1\leq\alpha,\beta\leq n}} A^{\alpha,\beta}_{i,j}(t,x)\partial_\beta\varphi^j(x)\overline{\partial_\alpha\varphi^i}(x).$$

L'espace de Sobolev homogène $\dot{H}^1(\mathbb{R}^n; \mathbb{C}^m)$ est la fermeture de $C^\infty_{\mathsf{c}}(\mathbb{R}^n; \mathbb{C}^m)$ pour la semi-norme $\|\nabla\varphi\|_{L^2(\mathbb{R}^n;\mathbb{C}^m)}$.

La troisième partie de cette thèse est consacrée à l'équation de Navier–Stokes incompressible. En exploitant la théorie des espaces de tentes, nous résolvons un problème ouvert depuis long temps, relatif à la continuité temporelle et au comportement en grand temps des solutions mild dans l'espace de Koch–Tataru, avec de données initiales appartenant à l'espace critique extrême BMO^{-1}.

2.2 État de l'art

Afin de mieux appréhender le rôle des espaces de tentes, nous proposons d'abord un bref exposé de deux théories classiques de bien-posé des problèmes linéaires de Cauchy paraboliques sous forme (1.1): la théorie de la régularité maximale L^p et la théorie de la régularité maximale de Lions. En suite, nous introduisons la théorie fondée sur les espaces de tentes.

2.2.1 Régularité maximale L^p

Cette théorie est principalement spécialisée dans le cas $F = 0$. Introduisons d'abord le cas autonome $A = A(x)$, c'est-à-dire que les coefficients sont indépendants du temps. On définit alors l'opérateur

$$L := -\operatorname{div}(A(x)\nabla) \tag{2.3}$$

sur $L^2(\mathbb{R}^n)$ avec le domaine

$$D(L) := \{\varphi \in L^2(\mathbb{R}^n) : \operatorname{div}(A(x)\nabla\varphi) \in L^2(\mathbb{R}^n)\}.$$

Grâce à la théorie des opérateurs accrétifs maximaux, on sait que $-L$ engendre un semi-groupe analytique sur l'espace de Banach $X = L^2(\mathbb{R}^n)$, noté $(e^{-tL})_{t\geq 0}$. On peut donc interpréter le problème de Cauchy (2.1) comme un problème de Cauchy abstrait sur X de la forme

$$\begin{cases} \partial_t u + Lu = f & \text{dans } X, \quad 0 < t < T \\ u(0) = 0 \end{cases} \tag{2.4}$$

et adopter la notion de *solution mild*. Plus précisément, pour $0 < T < \infty$ et $f \in L^1((0,T);X)$, on dit que $u \in L^1((0,T);X)$ est une solution mild de (2.4) si, pour tout $t \in [0,T]$ et tout $x^* \in D(L^*) \subset X^*$,

$$\langle x^*, u(t)\rangle + \int_0^t \langle L^* x^*, u(s)\rangle ds = \int_0^t \langle x^*, f(s)\rangle ds.$$

Les paires désignent l'action de l'espace dual X^* sur X. On montre en effet qu'il existe une unique solution mild de (2.4), donnée par *l'opérateur de Duhamel*

$$\mathcal{L}_1(f)(t) := \int_0^t e^{-(t-s)L} f(s) ds, \quad t > 0. \tag{2.5}$$

Les intégrales de (2.5) s'interprètent comme des intégrales de Bochner à valeurs dans X.

Pour prendre en compte les données initiales, on peut rechercher un exposant $p \in (1,\infty)$ de sorte que pour tout $f \in L^p((0,T);X)$, la solution mild u (fournie par l'opérateur de Duhamel) satisfasse aux *estimations de régularité maximale* L^p:

$$\|\partial_t u\|_{L^p((0,T);X)} + \|Lu\|_{L^p((0,T);X)} \lesssim \|f\|_{L^p((0,T);X)}. \tag{2.6}$$

Si ces estimations sont vérifiées, alors u est continue en temps, à valeurs dans l'espace d'interpolation réelle $Y = (D(L), X)_{1/p,p}$. On peut donc prendre comme donnée initiale $u_0 \in Y$.

Cette stratégie s'inscrit dans un cadre théorique général et s'applique à tout générateur $-L$ d'un semi-groupe analytique sur un espace de Banach X (voir, par exemple, le livre de Ladyženskaja, Solonnikov et Ural'ceva [LSU68]). Une approche pour établir l'estimation (2.6) pour $1 < p < \infty$ consiste à montrer que *l'opérateur de régularité maximale* $\mathcal{L}_0$, initialement défini sur $L^2((0,T); D(L))$ par

$$\mathcal{L}_0(f)(t) := \int_0^t Le^{-(t-s)L} f(s) ds, \quad t > 0, \tag{2.7}$$

se prolonge en un opérateur borné sur $L^p((0,T); X)$. Il s'agit d'un opérateur d'intégrale singulière, puisque la norme $\|Le^{-(t-s)L}\|_{\mathscr{L}(L^2(\mathbb{R}^n))}$ est de l'ordre de $|t-s|^{-1}$. Dès lors, des techniques d'analyse harmonique deviennent pertinentes. Pour les espaces de Hilbert $X = \mathfrak{H}$, De Simon [dS64] a d'abord employé des multiplicateurs de Fourier pour prouver que, pour tout générateur $-L$ d'un semi-groupe analytique sur $\mathfrak{H}$, l'estimation (2.6) est valide pour tout $p \in (1, \infty)$. Dans un contexte plus général d'espaces de Banach UMD, la caractérisation de la régularité maximale L^p a été obtenue par Weis [Wei01] en utilisant la R-bornitude du semi-groupe (notion introduite d'après les travaux de Bourgain [Bou86]). On renvoie le lecteur à la synthèse [KW04] pour vue d'ensemble complet.

Pour le cas non-autonome (coefficients dépendant du temps), on peut également définir $L(t) := -\operatorname{div}(A(t)\nabla)$ comme opérateur sur X pour presque tout $t > 0$ et et chercher à établir l'estimation (2.6). Toutefois, la caractérisation de la régularité maximale L^p pour des coefficients $A(t,x)$ dépendants du temps reste largement ouverte. Les résultats positifs existants exigent des hypothèses supplémentaires de régularité de Sobolev, même dans le cas le plus simple $X = L^2(\mathbb{R}^n)$ et $p = 2$; voir [DZ17, Fac18, AO19]). On renvoie également le lecteur au [HvNVW23, Chapter 17] pour l'état de l'art.

Par ailleurs, le choix des exposants q proches de 2 pour lesquels la théorie s'applique à l'espace de Banach $X = L^q(\mathbb{R}^n)$ via la R-bornitude est très limité ; voir [BK03]). En effet, il a été montré dans [HMM11] que pour tout $q < \frac{2n}{n+2}$ (resp. $r > \frac{2n}{n-2}$), il existe une matrice *indépendante* du temps $A = A(x)$ telle que le semi-groupe (e^{-tL}) n'est pas nécessairement borné sur $L^q(\mathbb{R}^n)$ (resp. $L^r(\mathbb{R}^n)$).

Troisièmement, comme mentionné ci-dessus, les données initiales compatibles avec cette théorie doivent être choisies dans l'espace d'interpolation réelle $Y = (D(L), X)_{1/p,p}$. Pour le Laplacien $L = -\Delta$, lorsque $X = L^q(\mathbb{R}^n)$, alors Y s'identifie à l'espace de Besov inhomogène $B^{2(1-1/p)}_{q,p}$. Cela exclut de nombreuses données initiales irrégulières appartenant à des espaces (homogènes) de fonctions ou de distributions particulièrement importants en EDP, y compris les espaces de Lebesgue.

2.2.2 Régularité maximale de Lions

La seconde théorie repose sur la notion de solutions faibles (d'énergie). On dit qu'une fonction $u \in L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}}(\mathbb{R}^n))$ est une *solution faible* de l'équation

$$\partial_t u - \operatorname{div}(A(t,x)\nabla u) = f + \operatorname{div} F,$$

si pour toute fonction test $\phi \in C^\infty_{\mathsf{c}}((0,\infty)\times\mathbb{R}^n)$,

$$-\int_{(0,\infty)\times\mathbb{R}^n} u\ \partial_t\phi + \int_{(0,\infty)\times\mathbb{R}^n} (A(t,x)\nabla u)\cdot\nabla\phi = (f,\phi) - (F,\nabla\phi).$$

Les paires à droite désignent le dual des distributions agissant sur les fonctions tests sur $(0,\infty)\times\mathbb{R}^n$. En vertu de la condition initiale $u(0)=u_0$, on impose que $u(t)$ converge vers u_0 dans $\mathscr{D}'(\mathbb{R}^n)$ lorsque $t\to 0$.

C'est J.-L. Lions [Lio57] qui a établi pour la première fois la théorie L^2 des solutions faibles. Plus précisément, pour $0<T<\infty$, $u_0\in L^2(\mathbb{R}^n)$, $f=0$ et $F\in L^2((0,T)\times\mathbb{R}^n)$, il existe une solution faible u dans l'espace d'énergie

$$L^\infty((0,T);L^2(\mathbb{R}^n))\cap L^2((0,T);H^1(\mathbb{R}^n)) \tag{2.8}$$

du problème de Cauchy

$$\begin{cases}\partial_t u - \operatorname{div}(A(t,x)\nabla u) = \operatorname{div} F, & 0<t<T,\ x\in\mathbb{R}^n\\ u(0)=u_0\end{cases},$$

qui satisfait en outre l'inégalité d'énergie

$$\sup_{0\le t\le T}\|u(t)\|_{L^2(\mathbb{R}^n)} + \|\nabla u\|_{L^2((0,T)\times\mathbb{R}^n)} \lesssim \|u_0\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^2((0,T)\times\mathbb{R}^n)}.$$

Notons qu'aucune hypothèse de régularité ou de symétrie sur les coefficients n'est requise, et que cette théorie admet des données initiales irrégulières dans $L^2(\mathbb{R}^n)$.

De plus, Lions a également démontré que l'espace

$$\left\{u\in L^2((0,T);H^1(\mathbb{R}^n)) : \partial_t u\in L^2((0,T);H^{-1}(\mathbb{R}^n))\right\}$$

s'injecte continûment dans $C([0,T];L^2(\mathbb{R}^n))$ (*théorème d'injection de Lions*). Par conséquent, toute solution faible dans l'espace d'énergie de l'équation $\partial_t u - \operatorname{div}(A\nabla u)=0$ possède une trace u_0 dans $L^2(\mathbb{R}^n)$.

Pour des coefficients réels, bien davantage est connu grâce au principe du maximum. En effet, celui-ci entraîne une régularité locale Hölderienne des solutions faibles, comme le montre la théorie de De Giorgi–Nash–Moser

[DG57, Nas58, Mos61, Mos64]. Dans ce contexte, la théorie L^2 complète revient à Aronson [Aro68, Aro67], qui a construit des solutions fondamentales $K(t,s;x,y)$ et établi que, pour tout $u_0 \in L^2(\mathbb{R}^n)$, l'intégrale

$$u(t,x) = (\Gamma_A(t,s)u_0)(x) = \int_{\mathbb{R}^n} K(t,s;x,y)u_0(y)dy$$

represent l'unique solution faible du problème de Cauchy homogène

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = 0, & s < t < T \\ u(s) = u_0 \end{cases}$$

dans l'espace d'énergie (2.8) vient de la théorie L^2 de Lions. De plus, $K(t,s;x,y)$ satisfait une décroissance gaussienne ponctuelle

$$0 \leq K(t,s;x,y) \lesssim \frac{1}{(t-s)^{n/2}} \exp\left(-c\frac{|x-y|^2}{t-s}\right), \tag{2.9}$$

pour un certain $c > 0$. Pour $0 \leq s < t < T$, l'opérateur $\Gamma_A(t,s)$ défini sur $L^2(\mathbb{R}^n)$ par

$$(\Gamma_A(t,s)u_0)(x) := \int_{\mathbb{R}^n} K(t,s;x,y)u_0(y)dy, \quad x \in \mathbb{R}^n$$

est appelé *propagateur associé à A*. On montre que $\Gamma_A(t,s)$ est une contraction dans $L^2(\mathbb{R}^n)$, en utilisant l'inégalité d'énergie.

Aronson a également traité des données initiales présentant une croissance de type « inverse-gaussien », c'est-à-dire $e^{-\alpha|x|^2}u_0 \in L^2(\mathbb{R}^n)$ pour un certain $\alpha > 0$. La classe de solutions considérée est alors caractérisée par

$$\int_0^T \int_{\mathbb{R}^n} e^{-\gamma|x|^2}|u(t,x)|^2 dtdx < \infty, \tag{2.10}$$

où $\gamma > 0$ dépend de A et α. Des termes sources de même nature (croissance inverse-gaussienne) ont aussi été étudiés.

Ces résultats établissent la bien-posédessité des solutions faibles pour des données initiales $u_0 \in L^p(\mathbb{R}^n)$ avec $2 \leq p \leq \infty$. Toutefois, contrairement aux conclusions de Lions, on ne peut en général pas recouvrer les traces dans $L^p(\mathbb{R}^n)$ (et donc la représentation intégrale des solutions) à partir de la classe trop vaste définie par Aronson.

Cette limitation incite à établir des estimations *a priori* plus fines, par exemple les estimations mixtes $L^p_t L^q_x$ pour les solutions faibles et leurs dérivées, en particulier pour la dérivée d'ordre le plus élevé ∇u. Toutefois, à notre connaissance, l'obtention d'estimations $L^p_t L^q_x$ pour des matrices de coefficients arbitraires $A(t,x)$ reste un problème ouvert. Des résultats partiels ont été obtenus sous des hypothèses supplémentaires. Par exemple, lorsque l'application

$x \mapsto A(t,x)$ est uniformément continue, le problème a été traité par Ladyženskaja, Solonnikov et Ural'ceva [LSU68]. Krylov, dans un travail ultérieur [Kry07b], a relaxé cette condition en ne demandant plus que $x \mapsto A(t,x)$ appartienne à VMO. Ces résultats restent valables pour des coefficients complexes et même pour des systèmes paraboliques, voir notamment les travaux de Dong et Kim [DK11a, DK11b, DK18].

La stratégie fondamentale à ces développements consiste d'abord à établir les estimations dans des cas simplifiés, tels que des coefficients constants ou indépendants de la variable spatiale, puis à les étendre à des configurations plus générales au moyen d'arguments de perturbation, pour lesquels les hypothèses de régularité des coefficients apparaissent inévitables.

Par ailleurs, dans le cas de coefficients complexes, tant le principe du maximum que la décroissance gaussienne des propagateurs (cf. (2.9)) ne fonctionnent pas, même dans le cadre autonome, voir [ACT96, AT98].

2.2.3 Espaces de tentes

Introduisons une théorie alternative possible fondée sur les *espaces de tente (pondérés)*, laquelle conduit à des estimations plus fines pour des coefficients irréguliers et des données initiales irrégulières. Les espaces de tente ont été introduits par Coifman, Meyer et Stein [CMS85] en 1985, et trouvent leur origine dans les travaux de Fefferman et Stein [FS72] sur les espaces de Hardy réels en 1972. Ces espaces offrent un cadre unifié pour l'étude de nombreuses familles d'espaces de fonctions sur $\mathbb{R}^n$ via leurs prolongements à l'espace demi-supérieur $(0,\infty) \times \mathbb{R}^n$. Comme nous le verrons, ils incluent les espaces L^p, les espaces de Hardy réels, ainsi que l'espace de moyenne oscillation bornée John–Nirenberg BMO.

Rappelons la définition des espaces de tente dans le contexte parabolique. Pour $\beta \in \mathbb{R}$ et $0 < p < \infty$, *l'espace de tentes (parabolique)* T^p_β est constitué des fonctions mesurables $u(t,y)$ sur $(0,\infty) \times \mathbb{R}^n$ telles que

$$\|u\|_{T^p_\beta} := \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \fint_{B(x,t^{1/2})} |t^{-\beta} u(t,y)|^2 dydt \right)^{p/2} dx \right)^{1/p} < \infty.$$

La forme de l'intégrale interne provient de la *fonctionnelle de Lusin*, dite aussi *fonction carrée conique*. Pour $p = \infty$, la norme est définie par la *fonctionnelle de Carleson* :

$$\|u\|_{T^\infty_\beta} := \sup_{B\text{: boules dans } \mathbb{R}^n} \left(\frac{1}{|B|} \int_0^{r(B)^2} \int_B |t^{-\beta} u(t,y)|^2 dydt \right)^{1/2}.$$

Nous verrons ci-après que l'exposant β en temps est directement lié à la régularité.

L'idée maîtresse réside dans le fait que ces normes exploitent d'abord une intégrabilité L^2 localisée à l'intérieur, puis s'appuient sur un contrôle non tangentiel pour accéder à la trace au bord (qui s'identifie à $\mathbb{R}^n$), où les espaces mixtes $L^p_t L^q_x$ classiques montrent leurs limites. Cette flexibilité rend les espaces de tente particulièrement adaptés à l'analyse des problèmes aux limites ou des problèmes aux données initiales.

Par ailleurs, la localisation intérieure convient idéalement à l'étude de la bornitude des opérateurs avec une décroissance L^2 localisée mais sans décroissance ponctuelle. Un exemple emblématique est le semi-groupe $(e^{-tL})_{t\geq 0}$ généré par l'opérateur elliptique $L = -\operatorname{div}(A(x)\nabla)$ défini en (2.3), à coefficients complexes seulement bornés et mesurables.

Ces propriétés ont ainsi alimenté, au cours des deux dernières décennies, des avancées remarquables dans les applications des espaces de tente (et de leurs variantes) aux EDP, notamment pour des équations elliptiques à coefficients rugueux ou à valeurs au bord irrégulières. Par exemple, Auscher et Egert [AE23a] exploitent largement ce cadre fertile pour établir la bien-posée des problèmes aux limites de Dirichlet, de régularité et de Neumann pour des systèmes elliptiques à structure en blocs, traitant les espaces de données au bord les plus vastes possibles, sans imposer la moindre condition de régularité ou de symétrie sur les coefficients.

2.2.4 Les espaces de tente rencontrent des équations paraboliques

À notre connaissance, la toute première utilisation des espaces de tente (sous la forme de mesures de Carleson) pour résoudre des équations d'évolution revient à Koch et Tataru [KT01], qui ont établi la bien-posée globale de l'équation de Navier–Stokes incompressible pour de petites données initiales dans l'espace critique extrême BMO^{-1}, résultat qui reste optimal à ce jour.

Pour l'étude des problèmes de Cauchy paraboliques de type divergence (2.1), les travaux pionniers d'Auscher, Monniaux et Portal [AMP12] ainsi que de Kriegler [AKMP12] ont d'abord examiné l'extension bornée de l'opérateur de régularité maximale $\mathcal{L}_0$ (pour des coefficients indépendants du temps) aux espaces de tente pondérés T^p_β. Un pas supplémentaire est franchi dans [AMP19], où ils établissent la bien-posée des solutions faibles d'énergie (au sens de la théorie de la régularité maximale de Lions) pour le *problème de Cauchy homogène* non-autonome

$$\begin{cases} \partial_t u - \operatorname{div}(A(t,x)\nabla u) = 0 \\ u(0) = u_0 \end{cases} \tag{2.11}$$

lorsque la donnée initiale u_0 est prise dans $L^p(\mathbb{R}^n)$ pour $2-\epsilon < p < \infty$, où $\epsilon > 0$ dépend de l'ellipticité de A. Dans le cas de coefficients réels, on peut prendre

$\epsilon = 1$. La matrice de coefficients $A(t,x)$ n'est supposée que uniformément elliptique, bornée, mesurable et à valeurs complexes.

Pour ce faire, ils réexaminent d'abord la théorie L^2 de Lions pour (2.11) et l'étendent au cas $T = \infty$ au sein de la classe de solutions vérifiant $\nabla u \in L^2(\mathbb{R}^{1+n}_+)$. En décalant l'instant initial, ils définissent les *propagateurs* $(\Gamma_A(t,s))_{0\leq s\leq t<\infty}$ comme une famille de contractions sur $L^2(\mathbb{R}^n)$ telle que, pour tout $s \geq 0$ et tout $h \in L^2(\mathbb{R}^n)$,

$$u(t,x) = (\Gamma_A(t,s)u_0)(x) \tag{2.12}$$

soit la solution faible unique sur $(s,\infty) \times \mathbb{R}^n$ du problème de Cauchy

$$\begin{cases} \partial_t u - \mathrm{div}_x(A(t,x)\nabla_x u) = 0, & (t,x) \in (s,\infty) \times \mathbb{R}^n \\ u(s) = h \end{cases}$$

avec $\nabla u \in L^2((s,\infty) \times \mathbb{R}^n)$. Cette définition coïncide, pour les coefficients réels, avec celle de solutions fondamentales due à Aronson, bien qu'il n'existe pas nécessairement de représentation par noyau.

Alors la solution faible u est construite par extension de *l'application de résolution par propagateur* $\mathcal{E}_A$, initialement définie de $L^2(\mathbb{R}^n)$ à $L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}}(\mathbb{R}^n))$ par

$$\mathcal{E}_A(u_0)(t,x) := (\Gamma_A(t,0)u_0)(x), \quad t > 0, \ x \in \mathbb{R}^n, \tag{2.13}$$

La classe de solutions est caractérisée par la norme de ∇u dans l'espace de tente T^p_0. [1] Plus précisément, pour $2-\epsilon < p < \infty$ et $u_0 \in L^p(\mathbb{R}^n)$, $u = \mathcal{E}_A(u_0)$ est une solution faible globale de (2.11) satisfaisant

$$\|\nabla u\|_{T^p_0} \eqsim \|u_0\|_{L^p(\mathbb{R}^n)}. \tag{2.14}$$

On remarque que, pour $p = 2$, l'espace de tente T^2_0 s'identifie à $L^2(\mathbb{R}^{1+n}_+)$, de sorte que cette équivalence se lit directement à partir de l'égalité d'énergie

$$\|u_0\|^2_{L^2(\mathbb{R}^n)} = 2\Re \int_0^\infty \int_{\mathbb{R}^n} (A(t,x)\nabla u(t,x)) \cdot \overline{\nabla u}(t,x) dt dx.$$

Un complément ultérieur par Zatoń [Zat20] a étendu l'équivalence ci-dessus au cas $p = \infty$:

$$\|\nabla u\|_{T^\infty_0} \eqsim \|u_0\|_{\mathrm{BMO}}.$$

Zatoń y établit également l'unicité et la représentation des solutions faibles dans la classe $\nabla u \in T^p_0$: pour $2 - \epsilon < p \leq \infty$, toute solution faible u de l'équation $\partial_t u - \mathrm{div}(A\nabla u) = 0$ avec $\nabla u \in T^p_0$ possède une trace (unique) $u_0 \in L^p(\mathbb{R}^n)$ (ou BMO si $p = \infty$), et s'écrit $u = \mathcal{E}_A(u_0)$, ou de façon équivalente

$$u(t) = \Gamma_A(t,0)u_0, \quad t > 0.$$

[1] Leur classe de solutions originale est donnée par $u \in X^p$ (*l'espace de Kenig–Pipher*, introduit dans [KP93]). Une théorie complète de bien-posée et de représentation y a été établie, y compris pour $p = \infty$ et $u_0 \in L^\infty(\mathbb{R}^n)$. Notamment, on a l'équivalence $\|u\|_{X^p} \eqsim \|u_0\|_{L^p(\mathbb{R}^n)}$ pour $2 - \epsilon < p \leq \infty$.

Zatoń a également étendu ces résultats aux systèmes paraboliques d'ordre supérieur.

Un travail très récent d'Auscher et Portal [AP25] a en outre considéré des termes sources $F \in T^p_0$ pour $2 \leq p \leq \infty$. Pour des coefficients réels, cette analyse a été étendue à $1 - \epsilon < p \leq \infty$.

Esquissons succinctement les principaux ingrédients des travaux [AMP19, Zat20, AP25]. Comme on l'a souligné dans le paragraphe final de la Section 2.2.2, la décroissance gaussienne ponctuelle ne fonctionnent pas pour les propagateurs associés à des coefficients complexes généraux. En revanche, dans [AMP19], on établit une décroissance gaussienne localisée en L^2, aussi appelées *estimations hors-diagonale* $L^2 - L^2$ ou *estimations de Davies–Gaffney*, pour la famille de propagateurs $(\Gamma_A(t,s))$. Plus précisément, il existe une constante $c > 0$ telle que, pour tous $E, F \subset \mathbb{R}^n$ boréliens et tout $f \in L^2(\mathbb{R}^n)$,

$$\|\mathbb{1}_E \Gamma_A(t,s) \mathbb{1}_F f\|_{L^2(\mathbb{R}^n)} \lesssim \exp\left(-c\frac{\operatorname{dist}(E,F)^2}{t-s}\right) \|\mathbb{1}_F f\|_{L^2(\mathbb{R}^n)}.$$

Ces estimations ont été introduites pour la première fois par Gaffney [Gaf59] et Davies [Dav92] pour le semi-groupe généré par l'opérateur de Laplace–Beltrami sur les variétés riemanniennes. Dans le cas autonome $A = A(x)$, les estimations hors-diagonale $L^2 - L^2$ pour le semi-groupe $(e^{-tL})_{t\geq 0}$ (généré par l'opérateur elliptique $L = -\operatorname{div}(A(x)\nabla)$ défini en (2.3)) ont été mises en évidence dans [AHL⁺02], jouant un rôle crucial dans la résolution de la conjecture de Kato.

La combinaison de cette décroissance localisée avec des estimations par fonctions carrées fournit des estimations *a priori* suffisantes pour établir l'existence de solutions faibles.

La preuve de l'unicité et de la représentation s'appuie sur une représentation intérieure des solutions faibles, que l'on peut voir comme un raffinement de l'identité de Green dans le cadre des coefficients irréguliers. Plus précisément, soit u une solution faible sur la bande (a,b) de l'équation $\partial_t u - \operatorname{div}(A(t,x)\nabla u) = 0$ telle que, pour un certain $\gamma \in (0, \frac{\Lambda_0^2}{16\Lambda_1^2(b-a)})$,

$$\int_{\mathbb{R}^n} \left(\int_a^b \int_{B(x,b^{1/2})} |u(t,y)|^2 dt dy \right)^{1/2} e^{-\gamma|x|^2} dx < \infty. \tag{2.15}$$

Alors u vérifie ce que nous nommons l'« *identité par homotopie* »

$$u(t) = \Gamma_A(t,s)u(s) \ \text{ dans } \mathscr{D}'(\mathbb{R}^n), \qquad \text{pour } a < s < t < b,$$

au sens où, pour tout $h \in C^\infty_c(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} u(t,x)\overline{h}(x)dx = \int_{\mathbb{R}^n} u(s,x)\overline{(\Gamma_A(t,s)^* h)}(x)dx. \tag{2.16}$$

On remarque que la condition ci-dessus rappelle la condition d'Aronson (2.10) (pour des coefficients réels), mais sans imposition de condition au bord. Dès lors, l'analyse se ramène à l'étude du comportement au bord des solutions et à la continuité des propagateurs dans des topologies compatibles.

2.2.5 Applications aux équations non-linéaires

Le modèle (2.1) joue un rôle important dans l'étude des EDP non-linéaires. Par exemple, il permet d'analyser l'existence et l'unicité de solutions pour des systèmes quasi-linéaires, au moyen d'arguments de linéarisation et de point-fixe, voir *e.g.*, [Lun95]. Certaines applications concernent également les problèmes à frontière libre [PS16, DHMT21] et les EDP stochastiques [AvNP14, PV19, AP25].

Un intérêt particulier émane de la mécanique des fluides, notamment des équations de Navier–Stokes. Par exemple, Monniaux [Mon99] a employé la régularité maximale L^p du Laplacien pour fournir une preuve alternative de l'unicité des solutions milds de l'équation de Navier–Stokes incompressible en 3D dans $C^0_t L^3_x$, résultat obtenu initialement par Furioli, Lemarié-Rieusset et Terraneo [FLRT00].

Plus récemment, Auscher et Frey [AF17] ont utilisé l'opérateur de régularité maximale du Laplacien sur les espaces de tente pour proposer une démonstration plus opératorielle du résultat de Koch et Tataru sur BMO^{-1} [KT01]. Un travail très récent de Danchin et Vasilyev [DV23] exploite également cet opérateur sur les espaces de tente pour étudier l'équation de Navier–Stokes incompressible inhomogène.

Motivés par des applications en analyse stochastique, Portal et Veraar posent le problème de la bien-posée du problème de Cauchy homogène (2.11) pour des données initiales appartenant aux espaces de Hardy–Sobolev homogènes $\dot{H}^{s,p}$, où s désigne la régularité et p l'intégrabilité. La définition précise de ces espaces est donnée en Section 5.1. Pour mémoire, et contrairement aux identifications usuelles (modulo polynômes), nous définissons $\dot{H}^{s,p}$ comme une réalisation, au sein des *distributions tempérées* $\mathscr{S}'(\mathbb{R}^n)$ de l'espace de Triebel–Lizorkin $\dot{F}^s_{p,2}$. Pour $s = 0$, cet espace est isomorphe à l'espace de Hardy $H^p(\mathbb{R}^n)$ si $p \le 1$, à l'espace de Lebesgue $L^p(\mathbb{R}^n)$ si $1 < p < \infty$, et à BMO si $p = \infty$. Pour $s < 0$, $\dot{H}^{s,\infty}$ est isomorphe à BMO^s introduit par Strichartz [Str80].

Problème 2.1 ([PV19, p.583])**.** Soit $A \in L^\infty((0,\infty) \times \mathbb{R}^n; \mathrm{Mat}_n(\mathbb{C}))$ uniformément elliptique. Existe-t-il un rang de $s \ge 0$ et de $p \in (1,\infty)$ tel que l'équivalence

$$\|u_0\|_{\dot{H}^{s,p}} \eqsim \|\nabla \mathcal{E}_A(u_0)\|_{T^p_{s/2}}$$

soit valide pour tout $u_0 \in \dot{H}^{s,p}$?

L'équivalence met clairement en correspondance la régularité des données initiales et le poids en temps dans la norme de l'espace de tente de la solution. Dans le cas particulier $A = \mathbb{I}$ (i.e. lorsque $\mathrm{div}(A\nabla)$ coïncide avec le Laplacien Δ), ils ont essayé de la démontrer pour tout $s \geq 0$ et tout $p \in (1, \infty)$, mais la preuve comporte une lacune. En fait, comme nous le verrons dans le Théorème 2.4, pour $s \geq 1$ et $1 \leq p \leq \infty$, l'équivalence $\|u_0\|_{\dot{H}^{s,p}} \eqsim \|\nabla e^{t\Delta} u_0\|_{T^p_{s/2}}$ n'est vraie que si et seulement si u_0 est une constante, auquel cas $\|u_0\|_{\dot{H}^{s,p}} = 0$.

Une autre motivation de notre étude de ce problème est que les espaces de Hardy–Sobolev homogènes $\dot{H}^{s,p}$ apparaissent naturellement dans les EDP non linéaires en tant qu'espaces invariants par changement d'échelle (ou dits critiques). Aborder cette question pourrait nous apporter des éclairages profonds sur une large classe d'EDP non linéaires. Par exemple, l'équation de Navier–Stokes incompressible admet une hiérarchie bien connue d'espaces critiques

$$\dot{H}^{\frac{n}{2}-1} \hookrightarrow L^n \hookrightarrow \dot{H}^{-1+\frac{n}{p},p} \hookrightarrow \mathrm{BMO}^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}, \quad 2 \leq n < p < \infty.$$

Ici, l'espace de Sobolev homogène $\dot{H}^{\frac{n}{2}-1}$ coïncide avec l'espace de Hardy–Sobolev homogène $\dot{H}^{\frac{n}{2}-1,2}$ (à constantes près), L^n coïncide avec $\dot{H}^{0,n}$, et BMO^{-1} coïncide avec $\dot{H}^{-1,\infty}$.

2.3 Résultats principaux

Cette section rassemble les résultats principaux de cette thèse. Commençons par l'équation de la chaleur, puis nous examinons successivement les problèmes de Cauchy paraboliques linéaires autonomes et non-autonomes. Enfin, nous présentons les résultats relatifs à l'équation de Navier–Stokes.

2.3.1 Équation de la chaleur

Un sujet fondamental dans l'étude de l'équation de la chaleur est l'investigation de la représentation des solutions de la chaleur :

Problème 2.2. Étant donnée une solution de l'équation de la chaleur u sur le demi-espace supérieur $(0, \infty) \times \mathbb{R}^n$ ou sur la bande $(0, T) \times \mathbb{R}^n$, quand peut-on affirmer que u admet une représentation par le semigroupe de la chaleur appliqué aux données initiales, c'est-à-dire

$$u(t) = e^{t\Delta} u_0 \tag{2.17}$$

pour un certain u_0 et pour tout $t \in (0, T)$?

Le cadre le plus général pour une telle représentation s'appuie sur les distributions tempérées. Plus précisément, étant donné $u_0 \in \mathscr{S}'(\mathbb{R}^n)$ (l'espace des distributions tempérées), la fonction $t \mapsto e^{t\Delta}u_0$ appartient à $C^\infty([0,\infty);\mathscr{S}'(\mathbb{R}^n))$. Réciproquement, il a été montré dans [Tay11, Chapter 3, Proposition 5.1] que toute $u \in C^\infty([0,\infty);\mathscr{S}'(\mathbb{R}^n))$ résolvant l'équation de la chaleur est représentée par le semigroupe de la chaleur appliqué à sa donnée initiale. Cet argument reste valable dans $C^1((0,\infty);\mathscr{S}'(\mathbb{R}^n)) \cap C([0,\infty);\mathscr{S}'(\mathbb{R}^n))$, ce qui semble clore la question. Cependant, il fait appel à la transformée de Fourier et ne se transpose pas à des équations plus générales (par exemple, des équations paraboliques à coefficients). On peut alors se demander si des informations plus concrètes, telles qu'une condition de croissance sur la solution, permettraient d'obtenir une représentation sans recourir à la transformée de Fourier. On observe toutefois que toute croissance dépassant l'inverse d'une gaussienne quand $|x| \to \infty$ est incompatible avec cette représentation.

Un autre cadre d'étude est celui des solutions non-négatives. Un résultat classique de Widder [Wid44, Theorem 6] montre qu'en dimension 1, toute solution C^2 non négative sur la bande doit être de la forme (2.17) pour une certaine mesure de Borel non-négative u_0. Krzyzanski [Krz64] a étendu ce résultat aux dimensions supérieures et aux solutions classiques d'équations paraboliques à coefficients réguliers, par une représentation interne et un argument de passage à la limite. Nous ferons également appel à cette idée ci-dessous, en cherchant toutefois à enlever l'hypothèse de positivité. Aronson a ensuite généralisé ce théorème aux solutions faibles non négatives d'équations paraboliques réelles (voir [Aro68, Theorem 11]).

La question de l'unicité est liée à la représentation mais présente d'enjeux distincts. Par exemple, mentionnons le travail pionnier sur la non-unicité de Tychonoff [Tyc35], ainsi que deux articles proposant des critères suffisants d'unicité sur des bandes. Celui de Täcklind [Täc36] fournit la condition de croissance ponctuelle optimale, tandis que celui de Gushchin [Gus82] donne une condition locale L^2 à croissance prescrite, également optimale mais plus souple pour des équations plus générales. Dans ces résultats, la croissance peut dépasser l'inverse d'une gaussienne lorsque $|x| \to \infty$, ce qui exclut alors le recours aux distributions tempérées : l'unicité peut tenir sans qu'il soit possible de représenter les solutions générales.

Avec ces observations, on se trouve face à deux approches très différentes pour aborder la représentation (et l'unicité) : l'une reposant exclusivement sur les distributions et la transformée de Fourier ; l'autre n'utilisant ces outils à aucun moment.

Notre premier théorème (voir Théorème 2.3) établit un lien entre ces deux cadres, c'est-à-dire qu'il permet d'obtenir des distributions tempérées, et pas seulement de simples fonctions mesurables ou mesures, comme données initiales, à partir de conditions d'intégrabilité locale. De telles conditions ne portent que sur l'intérieur, fournies par un contrôle uniforme.

Théorème 2.3 (Représentation des solutions de la chaleur)**.** *Soit* $0 < T \le \infty$.

Soit $u \in \mathscr{D}'((0,T) \times \mathbb{R}^n)$ *une solution distributionnelle de l'équation de la chaleur. Supposons que :*

(i) (Condition de taille) *Pour tout* $0 < a < b < T$*, il existe* $C(a,b) > 0$ *et* $\gamma \in (0, 1/4)$ *tels que, pour tout* $R > 0$*,*

$$\left(\int_a^b \int_{B(0,R)} |u(t,x)|^2 dtdx \right)^{1/2} \leq C(a,b) \exp\left(\frac{\gamma R^2}{b-a} \right);$$

(ii) (Contrôle uniforme) *Il existe une suite* (t_k) *tendant vers 0 telle que* $(u(t_k))$ *soit bornée dans l'espace des distributions tempérées* $\mathscr{S}'$*.*

Alors, il existe un unique $u_0 \in \mathscr{S}'$ *tel que* $u(t) = e^{t\Delta} u_0$ *pour tout* $0 < t < T$*.*

L'idée principale consiste à obtenir, à partir de la croissance en L^2 sur des rectangles, une représentation interne en semigroupe des fonctions caloriques : c'est-à-dire l'identité d'homotopie (2.16) pour $L = -\Delta$. Le problème de la représentation des solutions se ramène alors à l'étude de leur comportement aux limites en temps initial.

En appliquant ce résultat aux espaces de Hardy–Sobolev homogènes $\dot{H}^{s,p}$, on obtient une correspondance claire et robuste entre ces espaces et les espaces de tentes (voir Théorème 5.17 et Corollaire 5.19) :

Théorème 2.4 (Équation de la chaleur et espaces de tentes pondérés)**.** *Soient* $s \in \mathbb{R}$*,* $0 < p \leq \infty$ *et* $\varphi \in \mathscr{S}'(\mathbb{R}^n)$*.*

(i) (Estimations en espaces de tentes pondérés) *Supposons* $s < 1$*. Si* $g \in \dot{H}^{s,p}$*, alors la fonction* $(t,x) \mapsto \nabla e^{t\Delta} g(x)$ *appartient à* $T^p_{s/2}$ *et l'on a*

$$\|\nabla e^{t\Delta} g\|_{T^p_{s/2}} \eqsim \|g\|_{\dot{H}^{s,p}}.$$

(ii) (Représentation des solutions de la chaleur) *Soit* u *une solution distributionnelle de l'équation de la chaleur sur* $\mathbb{R}^{1+n}_+$ *telle que* $\nabla u \in T^p_{s/2}$*. Supposons* $s > -1$ *et* $\frac{n}{n+s+1} \leq p \leq \infty$*. Alors il existe un unique* $u_0 \in \mathscr{S}'$ *tel que* $u(t) = e^{t\Delta} u_0$ *pour tout* $t > 0$*.*

De plus :

(1) *Si* $s \geq 1$ *et* $\frac{n}{n+s-1} \leq p \leq \infty$*, alors* u *est une constante.*

(2) *Si* $-1 < s < 1$ *et* $\frac{n}{n+s+1} \leq p \leq \infty$*, alors il existe* $\varphi \in \dot{H}^{s,p}$ *et* $c \in \mathbb{C}$ *tels que* $u_0 = \varphi + c$*, d'où* $u(t) = e^{t\Delta}\varphi + c$ *pour tout* $t > 0$*.*

En conséquence, pour $-1 < s < 1$ *et* $\frac{n}{n+s+1} \leq p \leq \infty$*, l'application* $u_0 \mapsto (e^{t\Delta} u_0)(x) =: u(t,x)$ *est un isomorphisme de* $\dot{H}^{s,p} + \mathbb{C}$ *sur l'espace des solutions distributionnelles* u *de l'équation de la chaleur vérifiant* $\nabla u \in T^p_{s/2}$*, et l'on a*

$$\|u_0\|_{\dot{H}^{s,p}/\mathbb{C}} \eqsim \|\nabla u\|_{T^p_{s/2}}.$$

Le premier point (i) établit une correspondance précise entre la régularité Sobolev des données initiales et l'exposant du poids temporel choisi. Le second point (ii) montre que la condition $s < 1$ est essentiellement optimale.

Pour $s = 0$, le point (i) se lit comme une caractérisation des espaces de Hardy–Sobolev homogènes via les extensions caloriques. Cela rappelle le travail de Fefferman et Stein [FS72] sur la définition des espaces de Hardy à l'aide de fonctions carrées coniques dans le cadre parabolique, bien que leur approche ne fût pas liée à une équation particulière afin d'obtenir une définition intrinsèque des espaces de Hardy. Ici, nous nous limitons aux prolongements liés à l'équation de la chaleur. On peut également rapprocher ce résultat de la théorie de Littlewood–Paley, qui emploie plutôt des fonctions de carré verticales (voir [Tri20, §4.1]).

Notre méthode s'étend aussi aux espaces de Besov homogènes $\dot{B}^s_{p,p}$ pour les mêmes valeurs de s et p, en remplaçant les espaces de tentes par les espaces Z introduits par Barton et Mayboroda [BM16] et en appliquant l'interpolation réelle. Au point extrême $s = 1$ et $p = \infty$, cela fournit une nouvelle caractérisation des fonctions lipschitziennes (voir Proposition 5.24 et Théorème 5.25).

2.3.2 Problèmes de Cauchy paraboliques autonomes

Ensuite, considérons les problèmes de Cauchy paraboliques autonomes sous forme divergence

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(x)\nabla_x u) = f + \operatorname{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases} . \tag{2.18}$$

Supposons que la matrice de coefficients $A = A(x) \in L^\infty(\mathbb{R}^n; \mathrm{Mat}_n(\mathbb{C}))$ soit uniformément elliptique (cf. (2.2)). On donne

$$L := -\operatorname{div}(A(x)\nabla)$$

l'opérateur défini sur $L^2(\mathbb{R}^n)$ (voir (2.3)), et $(e^{-tL})_{t\geq 0}$ le semi-groupe analytique généré par $-L$ sur $L^2(\mathbb{R}^n)$.

2.3.2.1 Exposants critiques

Pour préciser nos résultats, introduisons d'abord quelques exposants critiques. Rappelons que la théorie L^p du semi-groupe (e^{-tL}) est gouvernée par quatre nombres critiques, définis dans [Aus07, Proposition 3.15] pour $1 < p < \infty$, puis étendus à $\frac{n}{n+1} < p < \infty$ dans [AE23a, §6] afin d'inclure les espaces de Hardy $H^p(\mathbb{R}^n)$. Ces nombres sont :

- $p_\pm(L) \in [\frac{n}{n+1}, \infty]$ tel que $(p_-(L), p_+(L))$ soit le plus grand ensemble ouvert d'exposants p pour lesquels le semi-groupe $(e^{-tL})_{t\geq 0}$ est uniformément borné sur $L^p(\mathbb{R}^n)$ pour $p > 1$ et sur $H^p(\mathbb{R}^n)$ pour $p \leq 1$;

- $q_\pm(L) \in [\frac{n}{n+1}, \infty]$ tel que $(q_-(L), q_+(L))$ soit le plus grand ensemble ouvert d'exposants p pour lesquels la famille $(t^{1/2}\nabla e^{-tL})_{t>0}$ est uniformément borné sur $L^p(\mathbb{R}^n)$ pour $p > 1$ et sur $H^p(\mathbb{R}^n)$ pour $p \leq 1$.

On sait que $p_-(L) = q_-(L) < \frac{2n}{n+2}$, $q_+(L) > 2$ et $p_+(L) \geq \frac{nq_+(L)}{n-q_+(L)}$[2]. Ces inégalités strictes sont optimales. On définit ensuite

$$p_-^\flat(L) := \max\{p_-(L), 1\}.$$

On dispose de la relation de dualité $p_+(L^*) = p_-^\flat(L)'$, où p' est le conjugué de Hölder de $p \in [1, \infty]$.

Nous étendons ces nombres critiques à notre cadre $\dot{H}^{s,p}$. Pour $-1 \leq s \leq 1$, définissons $p_\pm(s, L)$ par

$$\frac{1}{p_-(s,L)} := \begin{cases} \frac{1}{p_-(L)} + \frac{s}{n} & \text{si } 0 \leq s \leq 1 \\ \frac{1+s}{p_-(L)} - \frac{s}{q_+(L^*)'} & \text{si } -1 \leq s \leq 0 \end{cases},$$

et

$$p_+(s, L) := \max\{p_-(-s, L^*), 1\}'.$$

On vérifie que $p_\pm(0, L) = p_\pm(L)$, $p_-(-1, L) = q_+(L^*)' \in [1, 2)$, and $p_+(1, L) = q_+(L) \in (2, \infty]$.

Nous introduisons également plusieurs autres nombres qui paramétrent nos résultats. Pour plus de commodité, on introduit un paramètre β relié à l'exposant de régularité s par la relation

$$\boxed{s = 2\beta + 1}$$

avec $\beta > -1$ sans restriction supérieure. Pour $\beta > -1$, on définit les nombres $p_L(\beta) \in (0, 2)$ and $p_L^\flat(\beta) \in (0, 2)$ par

$$p_L(\beta) := \frac{np_-(L)}{n + (2\beta + 1)p_-(L)}, \quad p_L^\flat(\beta) := \frac{np_-^\flat(L)}{n + (2\beta + 1)p_-^\flat(L)},$$

qui coïncident dès que $p_-(L) \geq 1$. Posons enfin

$$\beta(L) := -\frac{1}{2} - \frac{n}{2}\left(\frac{1}{p_-(L)} - 1\right) \geq -1.$$

On vérifie que $\beta(L) \geq -1/2$ si et seulement si $p_-(L) \geq 1$. Nous introduisons ensuite l'exposant critique $\widetilde{p}_L(\beta) \in (0, 2)$ défini de la façon suivante :

(i) Si $p_-(L) \geq 1$, alors

$$\widetilde{p}_L(\beta) := \begin{cases} p_L(\beta) & \text{si } \beta \geq -1/2 \\ p_-(2\beta + 1, L) & \text{si } -1 < \beta < -1/2 \end{cases}.$$

[2] Par convention, on pose $p_+(L) = \infty$ si $q_+(L) \geq n$.

(ii) Si $p_-(L) < 1$, alors

$$\widetilde{p}_L(\beta) := \begin{cases} p_L(\beta) & \text{si } \beta \geq \beta(L) \\ \frac{(\beta(L)+1)q_+(L^*)+\beta-\beta(L)}{(\beta(L)+1)q_+(L^*)} & \text{si } -1 < \beta < \beta(L) \end{cases}.$$

Remarquons que $\widetilde{p}_L(\beta(L)) = 1$ et $\widetilde{p}_L(-1) = q_+(L^*)' = p_-(-1, L)$.

En particulier, pour le Laplacien négatif $L = -\Delta$, ces nombres s'expriment explicitement. On a

$$\begin{cases} p_-(-\Delta) = q_-(-\Delta) = \frac{n}{n+1}, \quad p_-^\flat(\Delta) = 1, \\ p_+(-\Delta) = q_+(-\Delta) = \infty, \end{cases}$$

et donc

$$p_-(s, -\Delta) = \frac{n}{n+s+1}, \quad p_+(s, -\Delta) = \infty, \quad -1 \leq s \leq 1.$$

Il s'ensuit

$$\widetilde{p}_{-\Delta}(\beta) = p_{-\Delta}(\beta) = \frac{n}{n+2\beta+2}, \quad p_{-\Delta}^\flat(\beta) = \frac{n}{n+2\beta+1}, \quad \beta > -1.$$

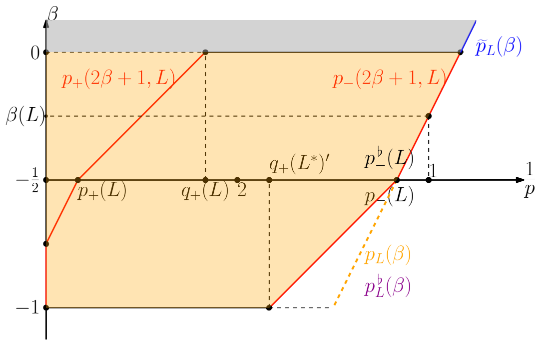
On observe également que, pour tout opérateur L,

$$\widetilde{p}_L(\beta) \geq p_L(\beta) \geq \frac{n}{n+2\beta+2} = \widetilde{p}_{-\Delta}(\beta).$$

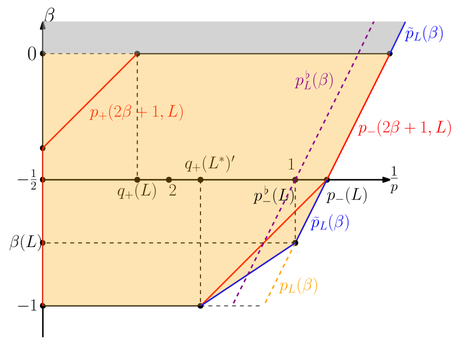
Pour illustrer ces exposants, nous proposons des représentations graphiques dans la figure 2.1, en distinguant les deux cas $p_-(L) \geq 1$ et $p_-(L) < 1$. Dans ces figures, nous remplaçons $1/p$ par p afin de faciliter la présentation. Lorsque $p < 2$, la courbe de $p_-(2\beta+1, L)$ est tracée en rouge, celle de $\widetilde{p}_L(\beta)$ en bleu, et celle de $p_L(\beta)$ en orange (ainsi que, le cas échéant, celle de $p_L^\flat(\beta)$ en violet lorsque $p_-(L) < 1$). Les trapèzes ombrés en orange délimitent les régions de bien-posé pour des données initiales dans $\dot{H}^{2\beta+1,p}$, tandis que la zone grisée correspond aux données initiales constantes.

Il est à noter que l'ensemble (plus restreint) délimité par les traits rouges (exclus) et les traits noirs (inclus) revêt une signification particulière et est appelé *rang d'identification* de l'opérateur L (voir Section 6.4, en particulier Proposition 6.36). [3]

[3] Dans le cas $p > 2$, les lignes brisées rouges représentant les valeurs de $p_+(2\beta+1, L)$ sont obtenues par symétrie par rapport au point $(1/p, \beta) = (1/2, -1/2)$ à partir de celles de $\max\{p_-(-2\beta-1, L^*), 1\}$. Elles dépendent donc de la valeur de $p_-(L^*)$. Dans la première (resp. seconde) illustration, cela correspond à $p_-(L^*) \geq 1$ (resp. $p_-(L^*) < 1$). Comme les valeurs de $p_-(L)$ et $p_-(L^*)$ sont indépendantes, on devrait en fait présenter quatre telles figures.



Cas $p_-(L) \geq 1$



Cas $p_-(L) < 1$

Figure 2.1: Exposants critiques pour les problèmes de Cauchy paraboliques autonomes

2.3.2.2 Résultats principaux

Le théorème suivant énonce le bien-posé des problèmes de Cauchy paraboliques autonomes de type (1.17) ; voir les théorèmes 6.11, 6.22 et 6.33 pour plus de détails.

Théorème 2.5 (Bien-posé des problèmes de Cauchy paraboliques autonomes)**.** *Soitent $\beta > -1$ et $\widetilde{p}_L(\beta) < p \leq \infty$. Soitent $\gamma > -1/2$ et $p_L^\flat(\gamma) < q \leq \infty$. On suppose*

$$\gamma \geq \beta, \quad 2\beta - \frac{n}{p} = 2\gamma - \frac{n}{q}.$$

(i) *Si $\beta < 0$, alors pour tout $u_0 \in \dot{H}^{2\beta+1,p}$, $F \in T^p_{\beta+1/2}$ et $f \in T^q_\gamma$, il existe une unique solution faible globale u du problème de Cauchy*

$$\begin{cases} \partial_t u - \operatorname{div}(A(x)\nabla u) = f + \operatorname{div} F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases},$$

telle que $\nabla u \in T^p_{\beta+1/2}$. De plus, on a l'estimation

$$\|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|u_0\|_{\dot{H}^{2\beta+1,p}} + \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^q_\gamma},$$

et $u \in C([0,\infty); \mathscr{S}')$. Si de plus $u_0 = 0$, alors u appartient également à $T^p_{\beta+1}$ et

$$\|u\|_{T^p_{\beta+1}} \lesssim \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^q_\gamma}.$$

(ii) *Si $\beta \geq 0$, alors l'énoncé précédent reste vrai à condition que les données initiales u_0 soient constantes (pour lesquelles $\|u_0\|_{\dot{H}^{2\beta+1,p}} = 0$).*

En effet, lorsque $\beta \geq 0$, comme nous le verrons au Théorème 2.7 (i), les données initiales constantes sont les seules compatibles avec la classe de solutions $\nabla u \in T^p_{\beta+1/2}$.

Quand $t \to 0$, la convergence de $u(t)$ vers u_0 se raffine dans des topologies plus fines que $\mathscr{S}'$, selon β et p. Considérons par exemple le problème de Cauchy homogène ($f = 0$, $F = 0$). Les solutions faibles sont construites en prolongeant l'*application de résolution par semi-groupe* $\mathcal{E}_L$, définie initialement de $L^2(\mathbb{R}^n)$ à $L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}}(\mathbb{R}^n))$ par

$$\mathcal{E}_L(u_0)(t,x) := (e^{-tL}u_0)(x). \tag{2.19}$$

Lorsque $(2\beta+1, p)$ appartient à le rang d'identification, cette extension préserve la régularité $\dot{H}^{2\beta+1,p}$, c'est-à-dire, pour tout $u_0 \in \dot{H}^{2\beta+1,p}$, [4]

$$\mathcal{E}_L(u_0) \in C_0([0,\infty); \dot{H}^{2\beta+1,p}) \cap C^\infty((0,\infty); \dot{H}^{2\beta+1,p}).$$

Dans ce cadre, pour $p \geq 1$ et $2\beta < n/p$, [5] cela implique aussi que $v = \mathcal{E}_L(u_0)$ est une solution forte du problème de Cauchy abstrait

$$\begin{cases} \partial_t v + L_{\beta,p} v = 0 & \text{dans } \dot{H}^{2\beta+1,p} \\ v(0) = v_0 \end{cases},$$

où $-L_{\beta,p}$ est le générateur du semi-groupe étendu à $\dot{H}^{2\beta+1,p}$. Remarquons que $p < 1$ est possible ici, même s'il n'existe, à notre connaissance, aucune théorie abstraite de solutions fortes dans les espaces quasi-Banach. En revanche, lorsque $(2\beta + 1, p)$ n'appartient pas à le rang d'identification, l'extension $\mathcal{E}_L$ ne se comporte plus comme un semi-groupe.

Pour les problèmes de Cauchy inhomogènes ($u_0 = 0$ et $F = 0$), les solutions faibles sont construites par l'extension de l'opérateur de Duhamel $\mathcal{L}_1$ défini en (2.5) à l'espace de tentes T^q_γ. En effet, dès que $\gamma > -1/2$ et $p^\flat_L(\gamma) < p \leq \infty$, pour tout $f \in T^q_\gamma$, la solution faible $u = \mathcal{L}_1(f)$ vérifie

$$\|u\|_{T^q_{\gamma+1}} + \|\nabla u\|_{T^q_{\gamma+1/2}} \lesssim \|f\|_{T^q_\gamma}.$$

En outre, tant $\partial_t u$ que $\operatorname{div}(A\nabla u)$ (au sens des distributions) appartiennent à T^q_γ, et l'on a

$$\|\partial_t u\|_{T^q_\gamma} + \|\operatorname{div}(A\nabla u)\|_{T^q_\gamma} \lesssim \|f\|_{T^q_\gamma}. \tag{2.20}$$

Nous appelons (2.20) l'estimation de régularité maximale sur les espaces de tentes. En effet, comme l'exposant γ tie à régularité, cette inégalité montre que $\partial_t u$ et $\operatorname{div}(A\nabla u)$ ne peuvent pas appartenir simultanément à un espace plus régulier ; sinon la régularité de f pourrait être augmentée. C'est exactement l'esprit de la régularité maximale, même si, ici, l'interprétation ne passe plus

[4] Ici, $C([0,\infty); E)$ désigne l'espace des fonctions continues tendant vers 0 lorsque $t \to \infty$ dans la topologie prescrite sur E.

[5] Lorsque $2\beta - n/p \geq 0$, l'espace $\dot{H}^{2\beta+1,p}$ n'est pas un espace de Banach, mais seulement semi-normé.

par la théorie des semigroupes comme dans la théorie de la régularité maximale L^p.

Pour établir ces estimations, nous développons une théorie des opérateurs d'intégrale singulière sur les espaces de tentes. L'exemple prototypique est l'opérateur de régularité maximale $\mathcal{L}_0$ défini en (2.7). C'est De Simon [dS64] qui a d'abord montré que $\mathcal{L}_0$ s'étend en un opérateur borné sur $L^2((0,\infty);L^2(\mathbb{R}^n)) \simeq T^2_0$. Auscher, Kriegler, Monniaux et Portal [AKMP12] ont ensuite étudié son extension bornée aux espaces de tentes T^p_β pour certains intervalles de β et p, avec bornes explicites.

Notre cadre affine la définition des opérateurs d'intégrale singulière en exploitant la décroissance hors-diagonale L^p–L^q, corrigeant ainsi une imprécision de [AKMP12]. Nous proposons par ailleurs une nouvelle méthode de preuve : la stratégie consiste à majorer, point par point, la fonction de carré conique (norme d'espace de tentes) par la *fonction de carré verticale* (*fonctionnelle de Littlewood–Paley–Stein*), puis à appliquer l'extrapolation de Rubio de Francia et l'interpolation de Stein. Cette approche améliore les résultats antérieurs ; en particulier, on obtient le résultat suivant (voir Proposition 6.17) :

Théorème 2.6 (Continuité de l'opérateur de régularité maximale sur les espaces de tentes)**.** *Soient $\gamma > -1/2$ et $p^\flat_L(\gamma) < p \le \infty$. Alors $\mathcal{L}_0$ s'étend en un opérateur borné sur T^q_γ.*

Pour $\gamma = 0$, on retrouve les résultats de Huang [Hua17] avec une démonstration plus simple.

Nous établissons également la représentation (et donc l'unicité) des solutions faibles dans cette classe. Pour plus de commodité, on pose $\mathcal{E}_L(c) = c$ pour toute fonction constante c.

Théorème 2.7 (Représentation des solutions faibles des équations paraboliques autonomes)**.** *Soient $\beta > -1$ et $\widetilde{p}_L(\beta) < p \le \infty$. Soit u une solution faible de l'équation*

$$\partial_t u - \operatorname{div}_x(A(x)\nabla_x u) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n,$$

telle que $\nabla u \in T^p_{\beta+1/2}$. Alors u possède une trace $u_0 \in \mathscr{S}'$ au sens où $u(t)$ converge vers u_0 dans $\mathscr{S}'$ quand $t \to 0$. De plus :

(i) *Si $\beta \ge 0$ et $\frac{n}{n+2\beta} \le p \le \infty$, alors u est constante.*

(ii) *Si $-1 < \beta < 0$, alors il existe $\varphi \in \dot{H}^{2\beta+1,p}$ et $c \in \mathbb{C}$ tels que $u_0 = \varphi + c$ et $u = \mathcal{E}_L(\varphi) + c$, où $\mathcal{E}_L$ désigne un prolongement approprié de la carte solution de semi-groupe donnée par (2.19).*

Par conséquent, pour $-1 < \beta < 0$ et $\widetilde{p}_L(\beta) < p \le \infty$, l'application $u_0 \mapsto \mathcal{E}_L(u_0) = u$ est un isomorphisme de $\dot{H}^{2\beta+1,p} + \mathbb{C}$ sur l'espace des solutions faibles globales de $\partial_t u - \operatorname{div}(A\nabla u) = 0$ vérifiant $\nabla u \in T^p_{\beta+1/2}$, et l'on a

$$\|u_0\|_{\dot{H}^{2\beta+1,p}/\mathbb{C}} \eqsim \|\nabla u\|_{T^p_{\beta+1/2}}.$$

Pour le cas extrême $\beta = -1$, nous démontrons également l'existence de solutions faibles u du problème de Cauchy homogène ($f = 0$, $F = 0$) pour des données initiales $u_0 \in \dot{H}^{-1,p}$ lorsque $p_-(-1, L) = q_+(L^*)' < p \leq \infty$ (voir Proposition 6.52). Mais le problème d'unicité et de représentation reste ouvert.

Des résultats analogues sont également établis pour des données initiales dans les espaces de Besov homogènes $\dot{B}^s_{p,p}$ avec les même valeurs de s et p.

2.3.3 Problèmes de Cauchy paraboliques non-autonomes

Considérons désormais les problèmes de Cauchy paraboliques non-autonomes sous forme (2.1)

$$\begin{cases} \partial_t u - \mathrm{div}_x(A(t,x)\nabla_x u) = f + \mathrm{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases},$$

où $A \in L^\infty((0,\infty) \times \mathbb{R}^n; \mathrm{Mat}_n(\mathbb{C}))$ est uniformément elliptique.

Commençons par introduire les exposants critiques. Soit $(\Gamma_A(t,s))_{0 \leq s \leq t < \infty}$ les propagateurs associée à A définis en (2.12). Soient $p_\pm(A) \in [1,\infty]$ tels que $(p_-(A), p_+(A))$ soit le plus grand ensemble ouvert d'exposants p pour lesquels la famille $(\Gamma_A(t,s))$ s'étend en une famille uniformément bornée d'opérateurs sur $L^p(\mathbb{R}^n)$. On sait que $p_-(A) < 2$ et $p_+(A) > 2$. Dans le cas de coefficients indépendants du temps, ces nombres coïncident respectivement avec $p_-^\flat(L)$ et $p_+(L)$.

Nous paramétrons également nos résultats par $\beta \in \mathbb{R}$, lié à l'exposant de régularité s par

$$\boxed{s = 2\beta + 1}.$$

Posons

$$p_A(\beta) := \frac{np_-(A)}{n + (2\beta+1)p_-(A)}.$$

Soit $\zeta < -1/2$ un paramètre de référence fixé. Définissons alors les exposants critiques $p_\zeta^\pm(\beta)$ par

$$p_\zeta^-(\beta) := \begin{cases} \frac{2(2\zeta+1)p_-(A)}{4(\zeta-\beta)+(2\beta+1)p_-(A)} & \text{si } \zeta \leq \beta < -1/2 \\ p_A(\beta) & \text{si } \beta \geq -1/2 \end{cases},$$

et

$$p_\zeta^+(\beta) := \begin{cases} \frac{2(2\zeta+1)}{2\beta+1} & \text{si } \zeta \leq \beta < -1/2 \\ \infty & \text{si } \beta \geq -1/2 \end{cases}.$$

Remarquons que $p_\zeta^-(\zeta) = p_\zeta^+(\zeta) = 2$. Afin d'illustrer ces exposants, nous proposons des représentations graphiques dans la figure 2.2. Dans cette figure, nous notons aussi p pour $1/p$. La courbe en pointillés rouge correspond à $p_\zeta^-(\beta)$ pour $\beta < -1/2$, la courbe rouge continue à $p_\zeta^+(\beta)$, et la courbe bleue en pointillés à $p_A(\beta)$. Des lignes parallèles à cette dernière représentent les plongements entre les espaces de Hardy–Sobolev et les espaces de tentes pondérés, descendant vers le bas du plan.

Nous introduirons un nouveau paramètre $\beta_A \in [-1, -1/2)$, ne dépendant que de l'ellipticité de A et de la dimension n, qui sert de borne inférieure à β. En prenant $\zeta = \beta_A$, le trapèze ombré en orange délimite la région de bien-posé pour des données initiales dans $\dot{H}^{2\beta+1,p}$, tandis que la zone bleue ombrée correspond aux données initiales constantes.

En particulier, considérons le triangle orange situé sous la droite $\beta = -1/2$. Nous l'avons décrit de façon qu'il soit au-dessus de deux segments. En réalité, il s'agit d'un artefact de notre formulation destinée à simplifier l'exposé : on pourrait énoncer que ce triangle orange est délimité par une courbe convexe passant par $(0, -1/2)$, $(1/2, \beta_A)$ et $(1/p_-(A), -1/2)$. La forme exacte de cette courbe nous échappe. En effet, pour $\beta < -1/2$, la ligne rouge continue représentant $p^+_{\beta_A}(\beta)$ résulte uniquement de notre choix de β_A. Si l'on prenait plutôt la borne inférieure sans qu'elle soit atteinte, cette ligne devrait être tracée en pointillés, bien que le point $(0, -1/2)$ reste toujours inclus.

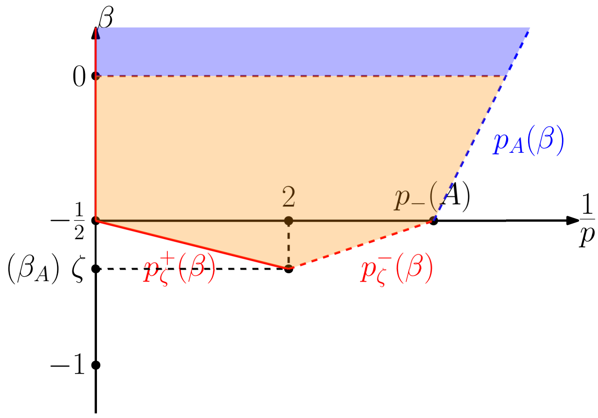


Figure 2.2: Exposants critiques pour les problèmes de Cauchy paraboliques non-autonomes

Cette limitation provient de l'absence d'exposants assurant la bornitude de $\nabla\Gamma_A(t,s)$ pour des s,t fixés, ce qui constitue une différence substantielle avec le cas autonome. Cela rend l'analyse beaucoup plus compliquée.

Passons à présent au bien-posé des problèmes de Cauchy paraboliques non-autonomes sous forme (2.1), comme annoncé.

Théorème 2.8 (Bien-posé des problèmes de Cauchy paraboliques non-autonomes)**.** *Il existe un paramètre $\beta_A \in [-1, -1/2)$, ne dépendant que de l'ellipticité de A et de la dimension n, tel que les propriétés suivantes soient satisfaites. Soient $\beta > \beta_A$ et $p^-_{\beta_A}(\beta) < p \le p^+_{\beta_A}(\beta)$. Soient $\gamma > -1/2$ et $p_A(\gamma) < q \le \infty$.*

Supposons

$$\gamma \geq \beta, \quad 2\gamma - \frac{n}{q} = 2\beta - \frac{n}{p}.$$

(i) *Si* $\beta < 0$*, alors pour tout* $u_0 \in \dot{H}^{2\beta+1,p}$*,* $F \in T^p_{\beta+1/2}$ *et* $f \in T^q_\gamma$*, il existe une unique solution faible globale* u *du problème de Cauchy*

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(t,x)\nabla_x u) = f + \operatorname{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases}$$

telle que $\nabla u \in T^p_{\beta+1/2}$*. De plus, on a*

$$\|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|u_0\|_{\dot{H}^{2\beta+1,p}} + \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^q_\gamma},$$

et $u \in C([0,\infty); \mathscr{S}')$*. Si de plus* $u_0 = 0$*, alors* $u \in T^p_{\beta+1}$ *et*

$$\|u\|_{T^p_{\beta+1}} \lesssim \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^q_\gamma}.$$

(ii) *Si* $\beta \geq 0$*, alors le même énoncé reste valide lorsque les données initiales* u_0 *sont constantes (pour lesquelles* $\|u_0\|_{\dot{H}^{2\beta+1,p}} = 0$*).*

Pour $\beta \geq 0$, les seules données initiales compatibles avec la classe de solutions $\nabla u \in T^p_{\beta+1/2}$ sont les constantes (voir Théorème 2.9 (i)).

Pour $\beta = -1/2$ ($s = 0$), on retrouve les résultats de [AMP19, Zat20] avec une preuve plus conceptuellement simple et opératorielle. On en déduit également ceux de [AP25] pour l'équation de Lions ($u_0 = 0$, $f = 0$) dans l'intervalle $p_-(A) < p < 2$.

Pour les problèmes inhomogènes ($u_0 = 0$, $F = 0$), les solutions faibles construites par l'extension de l'opérateur de Duhamel $\mathcal{L}^A_1$, initialement défini de $L^2(\mathbb{R}^{1+n}_+)$ à $L^\infty_{\mathsf{loc}}([0,\infty); L^2(\mathbb{R}^n))$ par l'intégrale de Bochner

$$\mathcal{L}^A_1(f)(t) := \int_0^t \Gamma_A(t,s) f(s) ds, \quad t > 0,$$

vers l'espace de tente T^q_γ. Nous montrons également que, dès que $\gamma > -1/2$ et $p_A(\gamma) < q \leq \infty$, pour tout $f \in T^q_\gamma$, la solution $u = \mathcal{L}^A_1(f)$ satisfait

$$\|u\|_{T^q_{\gamma+1}} + \|\nabla u\|_{T^q_{\gamma+1/2}} \lesssim \|f\|_{T^q_\gamma}.$$

La démonstration repose à nouveau sur la théorie des opérateurs d'intégrale singulière sur les espaces de tentes. En revanche, nous ne disposons pas, pour l'instant, d'estimations de régularité maximale pour les normes d'espace de tentes de $\partial_t u$ et $\operatorname{div}(A\nabla u)$, analogues à celles de (2.6), puisque nous n'avons pas d'estimations appropriées pour l'opérateur $\varphi \mapsto \operatorname{div}(A\nabla\Gamma_A(t,s)\varphi)$, du fait du manque d'analyticité des propagateurs.

Le théorème suivant présente la représentation des solutions faibles.

Theorem 2.9 (Représentation des solutions faibles des problèmes de Cauchy paraboliques non-autonomes)**.** *Soit β_A la constante fournie par le Théorème 2.8. Soient $\beta > \beta_A$ et $p_{\beta_A}^-(\beta) < p \leq p_{\beta_A}^+(\beta)$. Soit u une solution faible globale de l'équation $\partial_t u - \operatorname{div}(A\nabla u) = 0$ telle que $\nabla u \in T^p_{\beta+1/2}$. Alors u possède une trace $u_0 \in \mathscr{S}'$, au sens où $u(t)$ converge vers u_0 dans $\mathscr{S}'$ quand $t \to 0$. De plus :*

(i) *Si $\beta \geq 0$ et $\frac{n}{n+2\beta} \leq p \leq \infty$, alors u est une constante.*

(ii) *Si $\beta_A < \beta < 0$, alors il existe $\varphi \in \dot{H}^{2\beta+1,p}$ et $c \in \mathbb{C}$ tels que $u_0 = \varphi + c$ et $u = \mathcal{E}_A(\varphi) + c$, où $\mathcal{E}_A$ désigne l'extension de l'application de résolution par propagateurs définie en (2.13).*

Par conséquent, pour $\beta_A < \beta < 0$ et $p_{\beta_A}^-(\beta) < p \leq p_{\beta_A}^+(\beta)$, l'application $u_0 \mapsto \mathcal{E}_A(u_0) = u$ définit un isomorphisme de $\dot{H}^{2\beta+1,p} + \mathbb{C}$ sur l'espace des solutions faibles globales de $\partial_t u - \operatorname{div}(A\nabla u) = 0$ vérifiant $\nabla u \in T^p_{\beta+1/2}$, et l'on a

$$\|u_0\|_{\dot{H}^{2\beta+1,p}/\mathbb{C}} \eqsim \|\nabla u\|_{T^p_{\beta+1/2}}.$$

Cela résout enfin le Problème 2.1, y compris les rangs supplémentaires pour $s < 0$ ($\beta < -1/2$) et $p \leq 1$. Mentionnons en outre un corollaire d'application. Pour $p = 2$, on dispose de l'identification

$$T^2_\beta \simeq L^2\left((0,\infty), t^{-2\beta}dt;\ L^2(\mathbb{R}^n)\right) =: L^2_\beta(\mathbb{R}^{1+n}_+), \quad \beta \in \mathbb{R}.$$

Par conséquent, le Théorème 2.8 entraîne :

Corollaire 2.10. *Soit $\beta > -1/2$. Pour tout $f \in L^2_\beta(\mathbb{R}^{1+n}_+)$ et $F \in L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$, il existe une unique solution faible globale u du problème de Cauchy*

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(t,x)\nabla_x u) = f + \operatorname{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = 0 \end{cases}$$

telle que $\nabla u \in L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$. De plus, u appartient à $L^2_{\beta+1}(\mathbb{R}^{1+n}_+)$ et

$$\|u\|_{L^2_{\beta+1}(\mathbb{R}^{1+n}_+)} + \|\nabla u\|_{L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)} \lesssim \|F\|_{L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)} + \|f\|_{L^2_\beta(\mathbb{R}^{1+n}_+)}.$$

Remarquons que, pour $-1/2 < \beta < 1/2$, la fonction $\omega(t) := t^{-2\beta}$ appartient à la classe de Muckenhoupt A_p pour tout $p > -2\beta + 1$, en particulier, $\omega \in A_2$. Alors, ce corollaire fournit donc un exemple concret mais représentatif d'estimations pondérées en L^2 non couvertes par [DK18, Theorem 7.2], car il n'exige pas que $x \mapsto A(t,x)$ appartient à VMO.

2.3.4 Équations de Navier–Stokes

Considérons le problème de Cauchy pour l'équation de Navier–Stokes incompressible :

$$\begin{cases} \partial_t u + (u \cdot \nabla)u = \Delta u - \nabla p, \\ \nabla \cdot u = 0, \\ u(0) = u_0, \end{cases} \tag{NS}$$

où $u : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ désigne le champ de vitesses inconnu et $p : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ la pression scalaire inconnue.

Initiés par Fujita et Kato [FK64, Kat84], de nombreux travaux ont traité du bien-posé ou du non-bien-posé de l'équation de Navier–Stokes dans des espaces invariants par changement d'échelle (dits *critiques*), voir notamment [Che92, KP94, Can97, Pla98, BP08, Yon10, Wan15]. À ce jour, le meilleur résultat d'existence globale pour de petites données a été obtenu par Koch et Tataru [KT01] dans l'espace critique extrême BMO^{-1}, défini comme l'espace des divergences de champs vectoriels de BMO. Plus précisément, ils ont montré que, pour des données initiales $u_0 \in \mathrm{BMO}^{-1}$ de petite norme, il existe une solution mild globale u de (NS) associée à u_0 telle que

$$\|u\|_{X_\infty} := \sup_{t>0} \|t^{1/2} u(t)\|_{L^\infty(\mathbb{R}^n)} + \|u\|_{T_0^\infty} < \infty.$$

Depuis lors, l'un des enjeux majeurs est de comprendre la régularité de ces solutions de Koch–Tataru. Miura et Sawada [MS06], puis Germain, Pavlović et Staffilani [GPS07], ont établi l'analyticité spatiale de u. Malgré ces succès considérable, le problème de la régularité temporelle de u reste ouvert depuis longtemps. Le meilleur résultat à ce jour, d'après Auscher, Dubois et Tchamitchian [ADT04], affirme que toute solution mild $u \in X_\infty$ issue de $u_0 \in \mathrm{BMO}^{-1}$ appartient en fait à $L^\infty((0, \infty); \mathrm{BMO}^{-1})$.

Le théorème suivant résout ce problème tout en décrivant simultanément son comportement en grand temps (voir Théorèmes 8.1 et 8.2).

Théorème 2.11. *Soit $u_0 \in \mathrm{BMO}^{-1}$ à divergence nulle et $u \in X_\infty$ une solution mild de l'équation de Navier–Stokes associée à la donnée initiale u_0. Alors, u appartient à $C([0, \infty); \mathrm{BMO}^{-1})$ avec $u(0) = u_0$, et $u(t)$ tend vers 0 dans BMO^{-1} lorsque $t \to \infty$. Ici, l'espace BMO^{-1} est muni de la topologie faible* par rapport à l'espace de Hardy–Sobolev homogène $\dot{H}^{1,1}$.*

En fait, le même argument s'étend à toute bande $(0, T) \times \mathbb{R}^n$ pour $0 < T \leq \infty$.

Ce résultat est optimal. Pour la continuité, rappelons que le semi-groupe de la chaleur $(e^{t\Delta})$ n'est que faiblement* continu sur BMO^{-1}, si bien que la continuité faible* de $u(t)$ est le meilleur résultat envisageable. Pour le comportement en grand temps, il existe des solutions auto-similaires de petite norme en BMO^{-1} qui fournissent des contre-exemples montrant que la convergence ne peut pas être forte dans BMO^{-1}.

Nous soulignons qu'aucune hypothèse de petitesse n'est requise ni sur les données initiales, ni sur la solution elle-même. La démonstration repose exclusivement sur la dualité entre les espaces de tentes T^∞_0 et T^1_0. Les ingrédients principaux sont :

(i) la correspondance entre espaces de Hardy–Sobolev homogènes et espaces de tentes (cf. Théorème 2.4),

(ii) une nouvelle décomposition du noyau d'Oseen due à Auscher et Frey [AF17], utilisant l'opérateur de régularité maximale du laplacien sur les espaces de tentes.

2.4 Organisation, citations et notations

2.4.1 Organisation

L'organisation de cette thèse est la suivante.

Chapitre 3 rappel et affine de quelques propriétés fondamentales des espaces de tentes, qui seront utilisées de façon récurrente.

La première partie (Chapitre 4) est consacrée au développement de la théorie des opérateurs d'intégrale singulière sur les espaces de tentes. Les résultats principaux sont rassemblés en Section 4.2.1.

Dans la deuxième partie (Chapitres 5, 6 et 7), nous étudions les problèmes paraboliques linéaires de Cauchy sous forme (1.1). Le chapitre 5 est de l'équation de la chaleur.

Le chapitre 6 considère le cas autonome. Par linéarité, on décompose (2.1) en trois modèles : les problèmes de Cauchy inhomogènes ($u_0 = 0$, $F = 0$), l'équation de Lions ($u_0 = 0$, $f = 0$) et les problèmes de Cauchy homogènes ($f = 0$, $F = 0$). Les propriétés principales des solutions faibles sont énoncées respectivement en Théorème 6.11, Théorème 6.22 et Théorème 6.33. Section 6.6 contient une extension aux données initiales dans des espaces de Besov homogènes. Section 6.7 étude du problème de Cauchy homogène au point extrême de régularité $\beta = -1$.

Le chapitre 7 considère des cas non-autonomes. Les résultats principaux pour les problèmes inhomogènes, l'équation de Lions et les problèmes homogènes y sont exposés en Théorème 7.11, Théorème 7.15 et Théorème 7.26.

La dernière partie (Chapitre 8) est dédié aux applications à l'équation de Navier–Stokes incompressible.

2.4.2 Citations

Cette thèse est une compilation des publications et prépublications suivantes :

1. [AH24] *On representation of solutions to the heat equation*, avec Pascal Auscher, **Comptes Rendus Mathématique**, Vol.362 (2024), pp.761-768. DOI: 10.5802/crmath.593. arXiv: 2310.19330.
2. [AH25a] *On well-posedness and maximal regularity for parabolic Cauchy problems on weighted tent spaces*, avec Pascal Auscher, **Journal of Evolution Equations**, 25, 16 (2025). DOI: 10.1007/s00028-024-01041-x. arXiv: 2311.04844.
3. [AH25b] *On well-posedness for parabolic Cauchy problems of Lions type with rough initial data*, avec Pascal Auscher, **Mathematische Annalen** (2025). DOI: 10.1007/s00208-025-03149-y. arXiv: 2406.15775.
4. [Hou24] *On regularity of solutions to the Navier–Stokes equation with initial data in* BMO^{-1}. arXiv: 2410.16468.
5. [Hou25] *On well-posedness for non-autonomous parabolic Cauchy problems with rough initial data.* arXiv: 2505.09387.

2.4.3 Notations

Tout au long de cette thèse, pour tous $p, q \in (0, \infty]$, on pose

$$[p, q] := \frac{1}{p} - \frac{1}{q},$$

sous réserve qu'il n'y ait pas d'ambiguïté avec les intervalles fermés.

On écrira $X \lesssim Y$ (ou $X \lesssim_A Y$, resp.) pour signifier qu'il existe une constante C inessentielle (ou dépendant de A, resp.) telle que $X \leq CY$. On écrira $X \eqsim Y$ si $X \lesssim Y$ et $Y \lesssim X$.

Pour toute boule (euclidienne) $B \subset \mathbb{R}^n$, on note $r(B)$ son rayon.

Soit (X, μ) un espace mesuré. Pour tout sous-ensemble mesurable $E \subset X$ de mesure finie et strictement positive, et pour toute fonction $f \in L^1(E, \mu)$, on définit

$$\fint_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu.$$

On notera

$$\| \cdot \|_p := \| \cdot \|_{L^p(X,\mu)}.$$

On pose $\mathbb{R}^{1+n}_+ := \mathbb{R}_+ \times \mathbb{R}^n = (0, \infty) \times \mathbb{R}^n$. Pour toute fonction f définie sur $\mathbb{R}^{1+n}_+$, on désigne par $f(t)$ l'application $x \mapsto f(t, x)$ pour tout $t > 0$. Pour tout $\beta \in \mathbb{R}$ and tout $E \subset \mathbb{R}^{1+n}_+$, on définit

$$L^p_\beta(E) := L^p(E, t^{-p\beta} dtdy).$$

Le plus souvent, on omet de préciser la mesure de Lebesgue non pondérée dans l'intégrale, ainsi que le domaine dans la notation de l'espace de fonctions, lorsque le contexte le rend clair. On utilise la fonte sans-sérif c en indice des espaces de fonctions pour signifier « à support compact » sur l'ensemble prescrit, et l'abréviation loc lorsqu'une propriété tient sur tout compact du domaine donné.

Chapter 3

Tent spaces

> "*Il y a tout à faire en peinture. Je pense que la peinture, en un sens, ne fait que commencer.*"
>
> Claude Monet

Tent spaces were introduced by Coifman, Meyer and Stein [CMS85], originated from the work of Fefferman and Stein [FS72]. Distilled from vast literature dating back then, this chapter serves as a concise instruction for driving the machinery of tent spaces without delving into too many rigorous justifications and computations. The reader can refer to [CMS85] for the original definitions of tent spaces and the proofs omitted. Some needed refinements can be found in [AMR08, Aus11, HMM11, Ame14, Hua16, AA18, Ame18].

Definition 3.1 (Tent spaces)**.** Let $0 < p < \infty$, $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, and $\sigma \geq 0$. The *tent space* $T^{p;m}_\beta$ consists of (possibly vector-valued, strongly) measurable functions f on $\mathbb{R}^{1+n}_+$ for which

$$\|f\|_{T^{p;m}_\beta} := \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \fint_{B(x,t^{1/m})} |t^{-\beta} f(t,y)|^2 dtdy \right)^{p/2} dx \right)^{1/p} < \infty.$$

The tent space $T^{\infty;m}_{\beta,(\sigma)}$ consists of measurable functions f on $\mathbb{R}^{1+n}_+$ for which

$$\|f\|_{T^{\infty;m}_{\beta,(\sigma)}} := \sup_B \frac{1}{|B|^\sigma} \left(\int_0^{r(B)^m} \fint_B |t^{-\beta} f(t,y)|^2 dtdy \right)^{1/2} < \infty,$$

where B describes all the balls of $\mathbb{R}^n$. We call m the *homogeneity* and β the *weight* of the tent space.

Convention 3.2. In this thesis, with an emphasis on the parabolic settings (*i.e.*, the homogeneity $m = 2$), we abbreviate $T^{p;2}_\beta$ by T^p_β, $T^{\infty;2}_{\beta,(\sigma)}$ by $T^\infty_{\beta,(\sigma)}$, and $T^{\infty;2}_{\beta,(0)}$ by T^∞_β. We may also omit the script β if $\beta = 0$.

All the tent spaces are quasi-Banach spaces. For $1 \leq p < \infty$ and $\sigma \geq 0$, the spaces $T^{p;m}_\beta$ and $T^{\infty;m}_{\beta,(\sigma)}$ are Banach spaces.

3.1 Duality, density, and atomic decomposition

Recall that for any $p, q \in (0, \infty]$, we write $[p, q]$ for $\frac{1}{p} - \frac{1}{q}$, if there is no confusion with closed intervals.

By $L^2(\mathbb{R}^{1+n}_+)$-*duality*, we mean the $L^2(\mathbb{R}^{1+n}_+)$ inner product [1]

$$\langle f, g \rangle_{L^2(\mathbb{R}^{1+n}_+)} = \int_{\mathbb{R}^{1+n}_+} f(t,y)\overline{g}(t,y)dtdy.$$

Proposition 3.3 (Duality). *Let $m \in \mathbb{N}$ and $\beta \in \mathbb{R}$.*

(i) *For $1 \leq p < \infty$, the $L^2(\mathbb{R}^{1+n}_+)$-duality identifies the dual of $T^{p;m}_\beta$ with $T^{p';m}_{-\beta}$.*

(ii) *For $0 < p \leq 1$, it identifies the dual of $T^{p;m}_\beta$ with $T^{\infty;m}_{-\beta,([p,1])}$.*

Notice that for $p = 2$, the tent space $T^{2;m}_\beta$ identifies with $L^2_\beta(\mathbb{R}^{1+n}_+)$ by Fubini's theorem. Let $L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$ be the space of $L^2(\mathbb{R}^{1+n}_+)$-functions with compact support in $\mathbb{R}^{1+n}_+$.

Lemma 3.4 (Density). *Let $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, and $\sigma \geq 0$.*

(i) *The tent spaces $T^{p;m}_\beta$ and $T^{\infty;m}_{\beta,(\sigma)}$ embed into $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$.*

(ii) *For $0 < p < \infty$, the space $L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$ is dense in $T^{p;m}_\beta$.*

(iii) *For $0 < p \leq 1$, it is dense in $T^{\infty;m}_{-\beta,([p,1])}$ with respect to the weak* topology against elements of $T^{p;m}_\beta$ (or called weak* dense).*

Proof. The statements (i) and (ii) follow from truncation by compact subsets $K \subset \mathbb{R}^{1+n}_+$, see [Ame14, Lemma 3.3 and Proposition 3.5]. Let us prove (iii) with more details. Let $f \in T^{\infty;m}_{-\beta,([p,1])}$. For any $R > 1$, define $C_R := [1/R, R] \times B(0, R)$ and $f_R := f\mathbb{1}_{C_R}$. For any $g \in T^{p;m}_\beta$, note that

$$\left| \langle f - f_R, g \rangle_{L^2(\mathbb{R}^{1+n}_+)} \right| \leq \int_{(C_R)^c} |f||g| \lesssim \|f\|_{T^{\infty;m}_{-\beta,([p,1])}} \|\mathbb{1}_{(C_R)^c} g\|_{T^{p;m}_\beta},$$

which tends to 0 as $R \to \infty$ by dominated convergence. This proves the convergence of f_R to f as $R \to \infty$ for the weak* topology. □

[1]One can also adopt the one without conjugate on the test function. The results will not change.

For $0 < p \leq 1$, a measurable function a on $\mathbb{R}^{1+n}_+$ is called a $T^{p;m}_\beta$-*atom*, if there exists a ball $B \subset \mathbb{R}^n$ such that $\mathrm{supp}(a) \subset [0, r(B)^m] \times B$, and

$$\|a\|_{L^2_\beta(\mathbb{R}^{1+n}_+)} \leq |B|^{-[p,2]}.$$

Such a ball B is said to be *associated* to a.

Proposition 3.5 (Atomic decomposition)**.** *Let $\beta \in \mathbb{R}$ and $0 < p \leq 1$. For any $f \in T^{p;m}_\beta$, there exist $(\lambda_j)_{j\geq 1} \in \ell^p$ and a sequence of $T^{p;m}_\beta$-atoms $(a_j)_{j\geq 1}$ so that*

$$f = \sum_{j\geq 1} \lambda_j a_j,$$

and

$$\|f\|_{T^{p;m}_\beta} \eqsim \|(\lambda_j)\|_{\ell^p} = \left(\sum_{j\geq 1} |\lambda_j|^p\right)^{1/p}.$$

The notion of atomic decomposition provides an effective way to study the bounded extension of a linear operator by its action on atoms.

Corollary 3.6. *Let $\beta, \gamma \in \mathbb{R}$ and $0 < p \leq 1$. Let T be a bounded linear operator from $L^2_\beta(\mathbb{R}^{1+n}_+)$ to $L^2_\gamma(\mathbb{R}^{1+n}_+)$. If T is uniformly bounded from $T^{p;m}_\beta$-atoms to $T^{p;m}_\gamma$, then T extends to a bounded linear operator from $T^{p;m}_\beta$ to $T^{p;m}_\gamma$.*

3.2 Interpolation and Sobolev embedding

Let us first recall the definition of Z-spaces, as introduced by Barton and Mayboroda [BM16] but with a different notation.

Definition 3.7 (Z-spaces)**.** Let $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, and $0 < p < \infty$. The Z-*space* $Z^{p;m}_\beta$ consists of measurable functions f on $\mathbb{R}^{1+n}_+$ for which

$$\|f\|_{Z^{p;m}_\beta} := \left(\int_0^\infty \int_{\mathbb{R}^n} \left(\int_{t/2}^t \fint_{B(x,t^{1/m})} |s^{-\beta} f(s,y)|^2 ds dy\right)^{p/2} dx \frac{dt}{t}\right)^{1/p} < \infty.$$

For $p = \infty$, we set

$$\|f\|_{Z^{\infty;m}_\beta} := \sup_{t>0, x\in\mathbb{R}^n} \left(\int_{t/2}^t \fint_{B(x,t^{1/m})} |s^{-\beta} f(s,y)|^2 ds dy\right)^{1/2}.$$

Remark 3.8. For $\beta = 1/2$ and $p = \infty$, the space $Z^{\infty;m}_{1/2}$ agrees with the Kenig–Pipher space $X^{\infty;m}$ as introduced by [KP93]. More precisely, define *Kenig–Pipher non-tangential maximal function* by

$$\mathcal{N}_{*;m}(f)(x) := \sup_{t>0} \left(\int_{t/2}^{t} \fint_{B(x,t^{1/m})} |f(s,y)|^2 dy \frac{ds}{s} \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Let $0 < p \leq \infty$. The *Kenig–Pipher space* $X^{p;m}$ consists of measurable functions f on $\mathbb{R}^{1+n}_+$ for which

$$\|f\|_{X^{p;m}} := \|\mathcal{N}_{*;m}(f)\|_{L^p} < \infty.$$

Theorem 3.9 (Interpolation)**.** *Let $m \in \mathbb{N}$, $\beta_0, \beta_1 \in \mathbb{R}$, and $0 < p_0, p_1 \leq \infty$. Let $\theta \in (0,1)$ so that*

$$\beta = (1-\theta)\beta_0 + \theta\beta_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

(i) (Real interpolation) *The real interpolation space $(T^{p_0;m}_{\beta_0}, T^{p_1;m}_{\beta_1})_{\theta,p}$ identifies with*
$$\begin{cases} T^{p;m}_{\beta} & \text{if } \beta_0 = \beta_1 \\ Z^{p;m}_{\beta} & \text{if } \beta_0 \neq \beta_1 \end{cases}.$$

(ii) (Complex interpolation) *Let $0 < p_0 < p_1 \leq \infty$. Then the complex interpolation space $[T^{p_0;m}_{\beta_0}, T^{p_1;m}_{\beta_1}]_{\theta}$ identifies with $T^{p;m}_{\beta}$.*

We also recall the Sobolev embedding of tent spaces.

Proposition 3.10 (Sobolev embedding)**.** *Let $m \in \mathbb{N}$, $\beta_0 \in \mathbb{R}$, and $0 < p_0 < \infty$.*

(i) *Suppose $\beta_1 < \beta_0$ and $p_0 < p_1 \leq \infty$ with*
$$m\beta_0 - \frac{n}{p_0} = m\beta_1 - \frac{n}{p_1}.$$
Then $T^{p_0;m}_{\beta_0}$ embeds into $T^{p_1;m}_{\beta_1}$.

(ii) *Suppose $\beta_2 < \beta_0$ and $\sigma_2 \geq 0$ with*
$$m\beta_0 - \frac{n}{p_0} = m\beta_2 + n\sigma_2.$$
Then $T^{p_0;m}_{\beta_0}$ embeds into $T^{\infty;m}_{\beta_2,(\sigma_2)}$.

(iii) *Suppose $0 < p_0 < p_1 < p_2 \leq \infty$ and*
$$m\beta_0 - \frac{n}{p_0} = m\beta_1 - \frac{n}{p_1} = m\beta_2 - \frac{n}{p_2}.$$
Then it holds that
$$T^{p_0;m}_{\beta_0} \hookrightarrow Z^{p_1;m}_{\beta_1} \hookrightarrow T^{p_2;m}_{\beta_2}.$$

3.3 Equivalent characterizations

In this section, we present two equivalent characterizations of the tent space norm.

3.3.1 Carleson functional

First, define the *Carleson functional* by

$$\mathcal{C}(f)(x) := \sup_{x \in B} \left(\int_0^{r(B)^m} \fint_B |f(t,y)|^2 dtdy \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

where B describes balls in $\mathbb{R}^n$. Notice that

$$\|f\|_{T^{\infty;m}_\beta} = \|\mathcal{C}(t^{-\beta} f(t))\|_{L^\infty(\mathbb{R}^n)}.$$

Here and in the sequel, we write $\|f(t)\|_{T^p_\beta}$ for $\|f\|_{T^p_\beta}$ to emphasize the weight in the time variable.

Proposition 3.11 (A-C equivalence)**.** *Let $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, and $2 < p < \infty$. For any $f \in L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$, it holds that*

$$\|f\|_{T^{p;m}_\beta} \eqsim \|\mathcal{C}(t^{-\beta} f(t))\|_{L^p(\mathbb{R}^n)}.$$

3.3.2 Change of aperture

For any $\alpha > 0$ and $x \in \mathbb{R}^n$, we define the *cone* $\Gamma^m_\alpha(x)$ centered at x with *aperture* α and *homogeneity* m by

$$\Gamma^m_\alpha(x) := \{(t,y) \in \mathbb{R}^{1+n}_+ : |x-y| < \alpha t^{1/m}\}.$$

To illustrate cones of different apertures, we give graphic representations in Figure 3.1. We use blue curve for the cone $\Gamma^2_1(x)$ (with homogeneity $m=2$ and aperture $\alpha = 1$) and red for the cone $\Gamma^2_2(x)$ (with homogeneity $m=2$ and aperture $\alpha = 2$).

Define the *conical square function* by

$$\mathcal{A}^{(\alpha)}_{\beta;m}(f)(x) := \left(\int_0^\infty \fint_{B(x,\alpha t^{1/m})} |t^{-\beta} f(t,y)|^2 dtdy \right)^{1/2}, \; x \in \mathbb{R}^n. \tag{3.1}$$

Notice that for $0 < p < \infty$,

$$\|f\|_{T^{p;m}_\beta} \eqsim \|\mathcal{A}^{(1)}_{\beta;m}(f)\|_{L^p(\mathbb{R}^n)}.$$

In fact, changing aperture gives an equivalent norm.

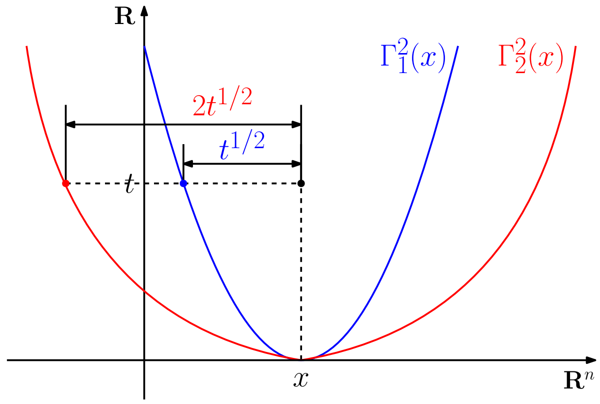


Figure 3.1: Cones with different apertures

Lemma 3.12 (Change of aperture)**.** *Let $m \in \mathbb{N}$ and $\beta \in \mathbb{R}$. For any $f \in L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$, it holds that*

$$\min\left\{1, \alpha^{n[p,2]}\right\} \|f\|_{T^{p;m}_\beta} \lesssim \|\mathcal{A}^{(\alpha)}_{\beta;m}(f)\|_{L^p} \lesssim \max\left\{1, \alpha^{n[p,2]}\right\} \|f\|_{T^{p;m}_\beta},$$

where $[p,2]$ denotes $\frac{1}{p} - \frac{1}{2}$, as we defined in the notation.

3.4 Two retractions

In this section, we introduce two retractions of tent spaces. For convenience, we restrict ourselves to Banach spaces. We say a Banach space B is a *retraction* of another Banach space A if there exist bounded linear operators $S \in \mathscr{L}(B, A)$ and $R \in \mathscr{L}(A, B)$ so that $R \circ S = \mathrm{Id}_B$. We also call R a *retraction* and S the *coretraction* associated to R.

3.4.1 Slice spaces

Slice spaces were introduced by Auscher and Mourgoglou [AM19, §3]. Let $m \in \mathbb{N}$, $0 < p \leq \infty$, and $\delta > 0$. The *slice space* $E^{p;m}_\delta$ consists of measurable (possibly $\mathbb{C}^n$-valued) functions f on $\mathbb{R}^n$ for which

$$\|f\|_{E^{p;m}_\delta} := \left\| \left(\fint_{B(\cdot,\delta^{1/m})} |f(y)|^2 \, dy \right)^{1/2} \right\|_{L^p(\mathbb{R}^n)} < \infty.$$

It is a quasi-Banach space, and for $1 \leq p \leq \infty$, it is a Banach space.

Lemma 3.13. *For any $\delta > 0$ and $1 \leq p \leq \infty$, $E^{p;m}_\delta$ is a retraction of $T^{p;m}_0$.*

Therefore, slice spaces inherit many properties of tent spaces. We list some useful ones below. By $L^2(\mathbb{R}^n)$-*duality*, we mean the $L^2(\mathbb{R}^n)$ inner product

$$\langle f, g\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} f(x)\overline{g}(x)dx.$$

(i) (Duality) Let $\delta > 0$ and $1 \leq p < \infty$. Then the $L^2(\mathbb{R}^n)$-duality identifies the dual of $E^{p;m}_\delta$ with $E^{p';m}_\delta$.

(ii) (Interpolation) Let $0 < p_0 < p < p_1 \leq \infty$ and $\theta \in (0,1)$ so that $1/p = (1-\theta)/p_0+\theta/p_1$. The complex interpolation space $[E^{p_0;m}_\delta, E^{p_1;m}_\delta]_\theta$ identifies with $E^{p;m}_\delta$, and so does the real interpolation space $(E^{p_0;m}_\delta, E^{p_1;m}_\delta)_{\theta,p}$.

(iii) (Change of aperture) Let $0 < p < \infty$ and $\delta, \delta' > 0$ be constants. Then for any $f \in L^2_{\mathsf{loc}}(\mathbb{R}^n)$, it holds that

$$\min\left\{1, \left(\frac{\delta}{\delta'}\right)^{\frac{n}{m}[p,2]}\right\} \|f\|_{E^{p;m}_{\delta'}} \lesssim \|f\|_{E^{p;m}_\delta} \lesssim \max\left\{1, \left(\frac{\delta}{\delta'}\right)^{\frac{n}{m}[p,2]}\right\} \|f\|_{E^p_{\delta'}}. \tag{3.2}$$

We also define the space $E^{1,p;m}_\delta$ as the collection of L^2_{loc}-functions g on $\mathbb{R}^n$ so that ∇g (in the sense of distributions) belongs to $E^{p;m}_\delta$, equipped with the semi-norm

$$\|g\|_{E^{1,p;m}_\delta} := \|\nabla g\|_{E^{p;m}_\delta}.$$

Also define $E^{-1,p;m}_\delta := \{g \in \mathscr{D}'(\mathbb{R}^n) : g = \operatorname{div}(G) \text{ for some } G \in E^{p;m}_\delta\}$ with the norm

$$\|g\|_{E^{-1,p;m}_\delta} := \inf\{\|G\|_{E^{p;m}_\delta} : G \in E^{p;m}_\delta,\ g = \operatorname{div} G\}.$$

The L^2-inner product on $\mathbb{R}^n$ realizes $E^{p';m}_\delta$ (resp. $E^{-1,p';m}_\delta$) as the dual of $E^{p;m}_\delta$ (resp. $E^{1,p;m}_\delta$) when $1 \leq p < \infty$.

3.4.2 Divergence of tent spaces

Let $m \in \mathbb{N}$, $\beta \in \mathbb{R}$, and $1 < p < \infty$. Let $\operatorname{div} T^{p;m}_\beta$ be the collection of distributions $f \in \mathscr{D}'(\mathbb{R}^{1+n}_+)$ of the form $f = \operatorname{div} F$ for some $F \in T^{p;m}_\beta$, endowed with the norm

$$\|f\|_{\operatorname{div} T^{p;m}_\beta} := \inf\left\{\|F\|_{T^{p;m}_\beta} : \operatorname{div} F = f,\ F \in T^{p;m}_\beta\right\}.$$

Denote by R the Riesz transform on $L^2(\mathbb{R}^{1+n}_+)$ given by

$$R(G)(t) = \nabla(-\Delta)^{-1/2}G(t), \quad t > 0.$$

We infer from [APA17, Theorem 2.4] that R is bounded on $T^{p;m}_\beta$. By duality, its dual operator R^*, given by

$$R^*(G)(t) := -(-\Delta)^{-1/2} \operatorname{div} G(t), \quad t > 0,$$

is also bounded on $T^{p;m}_\beta$. Let $f \in \operatorname{div} T^{p;m}_\beta$. Pick $F \in T^{p;m}_\beta$ with $\operatorname{div} F = f$. Define the operator $S : \operatorname{div} T^{p;m}_\beta \to T^{p;m}_\beta$ by

$$S(f) := RR^*(F) = -R(-\Delta)^{-1/2} \operatorname{div}(F).$$

The definition is independent of the choice of F. Using boundedness of R and R^*, we get $\|S(f)\|_{T^{p;m}_\beta} \lesssim \|F\|_{T^{p;m}_\beta}$. Taking infimum among all such F yields $\|S(f)\|_{T^{p;m}_\beta} \lesssim \|f\|_{\operatorname{div} T^{p;m}_\beta}$, so S is bounded.

Lemma 3.14. *Let $\beta \in \mathbb{R}$ and $1 < p < \infty$. Then $\operatorname{div} S = \mathrm{Id}_{\operatorname{div} T^{p;m}_\beta}$. Consequently, $\operatorname{div} T^{p;m}_\beta$ is a Banach space, and $\operatorname{div} : T^{p;m}_\beta \to \operatorname{div} T^{p;m}_\beta$ is a retraction with $S : \operatorname{div} T^{p;m}_\beta \to T^{p;m}_\beta$ as the coretraction.*

Proof. We only need to prove the identity. Let $f \in \operatorname{div} T^{p;m}_\beta$ and $F \in T^{p;m}_\beta$ with $\operatorname{div} F = f$. Pick $F_n \in L^2_{\mathsf{c}}((0,\infty); W^{1,2}_{\mathsf{c}}(\mathbb{R}^n))$ as a sequence approximating F in $T^{p;m}_\beta$. For any $t > 0$, we have

$$\operatorname{div} S(\operatorname{div} F_n)(t) = -\operatorname{div} \nabla(-\Delta)^{-1/2}(-\Delta)^{-1/2} \operatorname{div} F_n(t) = \operatorname{div} F_n(t)$$

in $L^2(\mathbb{R}^n)$, so

$$\operatorname{div} S(\operatorname{div} F_n) = \operatorname{div} F_n \quad \text{in } L^2(\mathbb{R}^{1+n}_+).$$

Ensured by boundedness of div and S, we take limits on both sides to get $\operatorname{div} S(f) = \operatorname{div} S(\operatorname{div} F) = \operatorname{div} F = f$ in $\operatorname{div} T^{p;m}_\beta$. □

Chapter 4

Singular integral operators on tent spaces

> "*... if you want to say something, you have to let the language itself say it, because language is usually more meaningful than the mere content that one wishes to convey.*"
>
> Elfriede Jelinek

The main objective of this chapter is to develop a framework of singular integral operators on tent spaces via the notion of off-diagonal decay. Section 4.1 introduces the definitions of two classes of singular integral operators, which are linked by duality. Section 4.2 is concerned with the extension of these two classes of singular integral operators to tent spaces. Main results are collected in Section 4.2.1.

This chapter contains the first half of the article "*On well-posedness and maximal regularity for parabolic Cauchy problems on weighted tent spaces*" [AH25a], written in collaboration with Pascal Auscher and published in *Journal of Evolution Equations*.

4.1 Singular integral operators

Let π_1, π_2 be the projections of $\mathbb{R}^{1+n} = \mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$ and $\mathbb{R}^n$ components, respectively. Denote by Δ the set $\{(t,s) \in (0,\infty) \times (0,\infty) : t = s\}$ and write $\Delta^c := ((0,\infty) \times (0,\infty)) \setminus \Delta$. We follow the notation $[p,q] := \frac{1}{p} - \frac{1}{q}$ for any $p, q \in (0,\infty]$, if there is no confusion with closed intervals.

Definition 4.1 (Off-diagonal decay)**.** Let $\kappa \in \mathbb{R}, m \in \mathbb{N}, 1 \leq q \leq r \leq \infty$, and $M > 0$ be constants. Let $\{K(t,s)\}_{(t,s)\in\Delta^c}$ be a family of bounded operators on $L^2(\mathbb{R}^n)$. We say that it has $L^q - L^r$ *off-diagonal decay of type* (κ, m, M) if there exists a constant $C > 0$ such that for any Borel sets $E, F \subset \mathbb{R}^n$, $f \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, and a.e. $(t,s) \in \Delta^c$,

$$\|\mathbb{1}_E K(t,s) \mathbb{1}_F f\|_{L^r} \leq C|t-s|^{-1+\kappa-\frac{n}{m}[q,r]} \left(1 + \frac{\operatorname{dist}(E,F)^m}{|t-s|}\right)^{-M} \|\mathbb{1}_F f\|_{L^q}. \tag{4.1}$$

4.1.1 Singular kernels

Let $\mathscr{L}(X)$ be the space of bounded linear operators on a (quasi-)Banach space X.

Definition 4.2 (Singular kernel). Let $\kappa \in \mathbb{R}, m \in \mathbb{N}, 1 \leq q \leq \infty$, and $M > 0$ be constants. We say that an operator-valued function $K : \Delta^c \to \mathscr{L}(L^2(\mathbb{R}^n))$ is a *singular kernel* (SK) *of type* (κ, m, q, M), if

1. K is strongly measurable;

2. There exists some constant $C > 0$ such that for a.e. $(t,s) \in \Delta^c$,

$$\|K(t,s)\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \leq C|t-s|^{-1+\kappa}; \tag{4.2}$$

3. The family $\{K(t,s)\}_{(t,s)\in\Delta^c}$ has off-diagonal $L^q - L^2$ decay of type (κ, m, M) if $1 \leq q \leq 2$, and off-diagonal $L^2 - L^q$ decay of type (κ, m, M) if $2 \leq q \leq \infty$.

Let $\mathrm{SK}^\kappa_{m,q,M}$ be the set of singular kernels of type (κ, m, q, M) and $\mathrm{SK}^\kappa_{m,q,\infty}$ be the intersection $\bigcap_{M>0} \mathrm{SK}^\kappa_{m,q,M}$.

Remark 4.3. The class $\mathrm{SK}^\kappa_{m,q,M}$ is stable with respect to truncation in the sense that, if $K \in \mathrm{SK}^\kappa_{m,q,M}$, then $\mathbb{1}_A K \in \mathrm{SK}^\kappa_{m,q,M}$ for any measurable set $A \subset \Delta^c$. The two cases, $A = \{t < s\}$ and $A = \{t > s\}$ are of particular interest.

Denote by T^* the adjoint of a linear operator T on $L^2(\mathbb{R}^n)$.

Lemma 4.4. *Let $1 \leq q \leq r \leq \infty$. Let $\{K(t,s)\}_{(t,s)\in\Delta^c}$ be a family of bounded operators on $L^2(\mathbb{R}^n)$ with $L^q - L^r$ off-diagonal decay of type (κ, m, M). Then $\{K(s,t)^*\}_{(t,s)\in\Delta^c}$ has $L^{r'} - L^{q'}$ off-diagonal decay of type (κ, m, M).*

Proof. Let $E, F \subset \mathbb{R}^n$ be two Borel sets. For any $f \in L^2(\mathbb{R}^n) \cap L^{r'}(\mathbb{R}^n)$ and a.e. $(t,s) \in \Delta^c$,

$$\begin{aligned}\|\mathbb{1}_F K(s,t)^*(\mathbb{1}_E f)\|_{q'} &= \sup_\phi \langle \mathbb{1}_F\phi, K(s,t)^*(\mathbb{1}_E f)\rangle_{L^2(\mathbb{R}^n)} \\ &= \sup_\phi \langle K(s,t)(\mathbb{1}_F\phi), \mathbb{1}_E f\rangle_{L^2(\mathbb{R}^n)} \\ &\le \sup_\phi \|\mathbb{1}_E K(s,t)(\mathbb{1}_F\phi)\|_r \|\mathbb{1}_E f\|_{r'} \\ &\lesssim |t-s|^{-1+\kappa-\frac{n}{m}[q,r]}\left(1+\frac{\operatorname{dist}(E,F)^m}{|t-s|}\right)^{-M}\|\mathbb{1}_E f\|_{r'},\end{aligned}$$

where ϕ is taken in $L^2(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)$ with support in F and $\|\phi\|_q = 1$. Note that $[q,r] = [r',q']$ to conclude the proof. □

Corollary 4.5. *If K belongs to $\mathrm{SK}^\kappa_{m,q,M}$, then $K^* : (t,s)\mapsto K(s,t)^*$ belongs to $\mathrm{SK}^\kappa_{m,q',M}$.*

Proof. The strong measurability of K^* follows from Pettis' measurability theorem (see [HvNVW16, Theorem 1.1.6]), since $L^2(\mathbb{R}^n)$ is separable. The boundedness condition is obvious, while the off-diagonal decay follows from Lemma 4.4. □

4.1.2 Integrals associated with singular kernels

For any function f on $\mathbb{R}^{1+n}_+$, denote by $f(s)$ the function $x\mapsto f(s,x)$. Let us first consider a non-singular integral. Let K be in $\mathrm{SK}^\kappa_{m,q,M}$ with $\kappa > 0$. For any $\beta > -1/2$ and $f \in L^2_\beta(\mathbb{R}^{1+n}_+)$, we claim that the integral

$$\int_0^t K(t,s)f(s)ds \tag{4.3}$$

makes sense as a Bochner integral on $L^2(\mathbb{R}^n)$ for a.e. $t\in(0,\infty)$. Indeed, (4.2) implies

$$\begin{aligned}&\left(\int_0^\infty\left(t^{-\beta-\kappa}\int_0^t \|K(t,s)f(s)\|_2\, ds\right)^2 dt\right)^{1/2} \\ &\quad\lesssim \left(\int_0^\infty\left(\int_0^t s^{\beta+\frac12}t^{-\beta-\kappa+\frac12}(t-s)^{-1+\kappa}\|s^{-\beta+\frac12}f(s)\|_2\frac{ds}{s}\right)^2\frac{dt}{t}\right)^{1/2}.\end{aligned}$$

Write $k(t,s) = \mathbb{1}_{\{s<t\}}(t,s)s^{\beta+\frac12}t^{-\beta-\kappa+\frac12}(t-s)^{-1+\kappa}$. For a.e. $t>0$,

$$\int_0^\infty k(t,s)\frac{ds}{s} = \int_0^1 \lambda^{\beta+\frac12}(1-\lambda)^{\kappa-1}\frac{d\lambda}{\lambda} \le C(\beta,\kappa).$$

With the same conditions, for a.e. $s>0$,

$$\int_0^\infty k(t,s)\frac{dt}{t} = \int_0^\infty \lambda^{\kappa-1}(\lambda+1)^{-\beta-\kappa-\frac12}d\lambda \le C(\beta,\kappa).$$

Schur test hence implies

$$\left(\int_0^\infty \left(t^{-\beta-\kappa}\int_0^t \|K(t,s)f(s)\|_2\, ds\right)^2 dt\right)^{1/2} \lesssim \|f\|_{L^2_\beta(\mathbb{R}^{1+n}_+)} < \infty.$$

This proves our claim. It also shows that the function

$$(t,y) \mapsto \left(\int_0^t K(t,s)f(s)ds\right)(y)$$

belongs to $L^2_{\beta+\kappa}(\mathbb{R}^{1+n}_+)$. Thanks to the strong measurability of K, we have the pointwise evaluation (see [HvNVW16, Proposition 1.2.25]) in the sense that for a.e. $(t,y) \in \mathbb{R}^{1+n}_+$,

$$\left(\int_0^t K(t,s)f(s)ds\right)(y) = \int_0^t (K(t,s)f(s))(y)ds. \tag{4.4}$$

For $f \in L^2_{-\beta-\kappa}(\mathbb{R}^{1+n}_+)$, by a duality argument, we can also define the $L^2(\mathbb{R}^n)$-valued Bochner integral

$$\int_t^\infty K(t,s)f(s)ds \tag{4.5}$$

for a.e. $t \in (0,\infty)$, and obtain similarly

$$\left(\int_t^\infty K(t,s)f(s)ds\right)(y) = \int_t^\infty (K(t,s)f(s))(y)ds \tag{4.6}$$

for a.e. $(t,y) \in \mathbb{R}^{1+n}_+$. We summarize the above discussion in the following lemma.

Lemma 4.6. *Let K be in $\mathrm{SK}^\kappa_{m,q,M}$ with $\kappa > 0$. For any $\beta > -1/2$,*

(i) *(4.3) defines a bounded operator from $L^2_\beta(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+\kappa}(\mathbb{R}^{1+n}_+)$, and (4.4) holds;*

(ii) *(4.5) defines a bounded operator from $L^2_{-\beta-\kappa}(\mathbb{R}^{1+n}_+)$ to $L^2_{-\beta}(\mathbb{R}^{1+n}_+)$, and (4.6) holds.*

We then turn to $\kappa \leq 0$, where a non-integrable singularity occurs at $t = s$. Still, as we are interested in handling functions of (t,x), we adopt a refined strategy based on the following lemma. A subset $Q \subset \mathbb{R}^{1+n}_+$ is called a *rectangle* if it is of product form $Q = \pi_1(Q) \times \pi_2(Q)$. It is called a *bounded* rectangle if both components are bounded.

Lemma 4.7. *Let $K \in \mathrm{SK}^\kappa_{m,q,M}$ with $\kappa \leq 0$ and $\frac{n}{m}|[q,2]| - \kappa < M \leq \infty$. Let Q_0, Q_1 be two bounded rectangles in $\mathbb{R}^{1+n}_+$ with*

$$\max\{\mathrm{dist}(\pi_i(Q_0), \pi_i(Q_1)) : i = 1,2\} \geq \epsilon > 0.$$

For a.e. $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$, *define*

$$\omega_{Q_0,Q_1}(t,s) := \|\mathbb{1}_{Q_0}(t)K(t,s)\mathbb{1}_{Q_1}(s)\|_{\mathscr{L}(L^2(\mathbb{R}^n))}.$$

Then, it holds that

$$\int_0^\infty \omega_{Q_0,Q_1}(t,s)ds \le C(\epsilon, Q_0, Q_1, \kappa, m, M, q),$$
$$\int_0^\infty \omega_{Q_0,Q_1}(t,s)dt \le C(\epsilon, Q_0, Q_1, \kappa, m, M, q).$$

Proof. By symmetry, it suffices to prove the first inequality. Recall that $\mathbb{1}_{Q_0}(t)(x) = \mathbb{1}_{\pi_1(Q_0)}(t)\mathbb{1}_{\pi_2(Q_0)}(x)$ and $\mathbb{1}_{Q_1}(s)(y) = \mathbb{1}_{\pi_1(Q_1)}(s)\mathbb{1}_{\pi_2(Q_1)}(y)$.

If $\operatorname{dist}(\pi_1(Q_0), \pi_1(Q_1)) \ge \epsilon/2$, for any $t \in \pi_1(Q_0)$, we have

$$\int_0^\infty \omega_{Q_0,Q_1}(t,s)ds \lesssim \int_{\pi_1(Q_1)} |t-s|^{-1+\kappa} ds \le (\epsilon/2)^{\kappa-1}|\pi_1(Q_1)|.$$

Otherwise, $\operatorname{dist}(\pi_1(Q_0), \pi_1(Q_1)) \le \epsilon/2$ but then $\operatorname{dist}(\pi_2(Q_0), \pi_2(Q_1)) \ge \epsilon$. Off-diagonal decay of $K(t,s)$ implies

$$\omega_{Q_0,Q_1}(t,s) \lesssim \begin{cases} \epsilon^{-mM}|t-s|^{-1+\kappa-\frac{n}{m}[q,2]+M}|\pi_2(Q_1)|^{[q,2]} & \text{if } 1 \le q \le 2 \\ \epsilon^{-mM}|t-s|^{-1+\kappa-\frac{n}{m}[2,q]+M}|\pi_2(Q_0)|^{[2,q]} & \text{if } 2 \le q \le \infty \end{cases}$$

by Hölder's inequality. In summary, we get

$$\int_0^\infty \omega_{Q_0,Q_1}(t,s)ds \lesssim_{Q_0,Q_1,q} \epsilon^{-mM} \int_{\pi_1(Q_1)} |t-s|^{-1+\kappa-\frac{n}{m}|[q,2]|+M} ds,$$

which converges as $M > \frac{n}{m}|[q,2]| - \kappa$. □

Let $L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$ be the subspace of $L^2(\mathbb{R}^{1+n}_+)$ consisting of functions with bounded support in $\overline{\mathbb{R}^{1+n}_+}$. For any $f \in L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$, define $\pi(f) := \pi_1(\operatorname{supp} f) \times \pi_2(\operatorname{supp} f)$ as a bounded rectangle. Let K, κ, and M be as in Lemma 4.7. For any bounded rectangle $Q \subset \mathbb{R}^{1+n}_+$ with $\operatorname{dist}(Q, \pi(f)) > 0$ and a.e. $t \in \pi_1(Q)$,

$$\int_0^\infty \mathbb{1}_{\pi_2(Q)}K(t,s)f(s)ds$$

is defined as a Bochner integral valued in $L^2(\mathbb{R}^n)$, since

$$\int_0^\infty \|\mathbb{1}_{\pi_2(Q)}(K(t,s)f(s))\|_2 \, ds \le \int_0^\infty \omega_{Q,\pi(f)}(t,s)\|f(s)\|_2 \, ds,$$

which is finite thanks to Lemma 4.7 and Schur test. Pointwise evaluation also holds, *i.e.*, for a.e. $(t,y) \in Q$,

$$\left(\int_0^\infty \mathbb{1}_{\pi_2(Q)}K(t,s)f(s)ds\right)(y) = \int_0^\infty \mathbb{1}_{\pi_2(Q)}(y)(K(t,s)f(s))(y)ds.$$

Furthermore, if $Q, Q' \subset (\mathbb{R}^{1+n}_+ \setminus \pi(f))$ are bounded rectangles with $Q \cap Q' \neq \varnothing$, then for a.e. $(t,y) \in Q \cap Q'$, we have

$$\int_0^\infty \mathbb{1}_{\pi_2(Q)}(y)(K(t,s)f(s))(y)dy = \int_0^\infty \mathbb{1}_{\pi_2(Q')}(y)(K(t,s)f(s))(y)ds. \tag{4.7}$$

Indeed, as $t \in \pi_1(Q) \cap \pi_1(Q')$, we have when $x \in \pi_2(Q) \cap \pi_2(Q')$

$$\text{LHS} = \int_0^\infty \mathbb{1}_{\pi_2(Q')}(y)\mathbb{1}_{\pi_2(Q)}(y)(K(t,s)f(s))(y)ds = \text{RHS}\,.$$

Thus, we can define the function

$$(t,y) \mapsto \int_0^\infty (K(t,s)f(s))(y)ds$$

almost everywhere on $\pi(f)^c$ as follows. Pick an *exhaustion* of $\pi(f)^c$ by bounded rectangles $\{Q_i\}_{i\in\mathbb{N}}$, *i.e.*, $\bigcup_i Q_i = \pi(f)^c$. Then, we set almost everywhere

$$\int_0^\infty (K(t,s)f(s))(y)ds := \int_0^\infty \mathbb{1}_{\pi_2(Q_i)}(y)(K(t,s)f(s))(y)ds, \tag{4.8}$$

if $(t,y) \in Q_i$. This definition makes sense almost everywhere and is clearly independent of the choice of exhaustion $\{Q_i\}_{i\in\mathbb{N}}$ by (4.7). We summarize the above discussion in the following lemma.

Lemma 4.8. *Let $\kappa \leq 0$, $\frac{n}{m}|[q,2]| - \kappa < M \leq \infty$, and $K \in \mathrm{SK}^\kappa_{m,q,M}$. Then for any $f \in L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$,*

$$(t,x) \mapsto \int_0^\infty (K(t,s)f(s))(x)ds$$

defines a function in $L^2_{\mathsf{loc}}(\overline{\mathbb{R}^{1+n}_+} \setminus \pi(f))$. Moreover, for any $g \in L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$ with $\pi(g) \cap \pi(f) = \varnothing$,

$$(t,s) \mapsto \langle K(t,s)f(s), g(t)\rangle_{L^2(\mathbb{R}^n)}$$

defines a function in $L^1((0,\infty)\times(0,\infty))$, and

$$\begin{aligned}\int_0^\infty \int_0^\infty &\langle K(t,s)f(s), g(t)\rangle_{L^2(\mathbb{R}^n)} dsdt \\ &= \int_0^\infty \int_{\mathbb{R}^n} \left(\int_0^\infty (K(t,s)f(s))(x)ds\right) \cdot \overline{g}(t,x)dtdx.\end{aligned}$$

Proof. The first part follows from (4.8), and the second part follows from the first part by writing

$$\langle K(t,s)f(s), g(t)\rangle_{L^2(\mathbb{R}^n)} = \langle \mathbb{1}_{\pi(g)}(t)K(t,s)\mathbb{1}_{\pi(f)}(s)f(s), g(t)\rangle_{L^2(\mathbb{R}^n)}$$

and using pointwise evaluation and Fubini's theorem. □

4.1.3 Singular integral operators

Define

$$M_{\kappa,q} := \begin{cases} 0 & \text{if } \kappa > 0 \\ \frac{n}{m}|[q,2]| - \kappa & \text{if } \kappa \leq 0 \end{cases}. \tag{4.9}$$

Definition 4.9 (SIO of type $+(\kappa, m, q, M)$)**.** Let $\kappa \in \mathbb{R}, m \in \mathbb{N}, 1 \leq q \leq \infty$, and $M_{\kappa,q} < M \leq \infty$ be constants. An operator T is called a *singular integral operator* (SIO) *of type* $+(\kappa, m, q, M)$, if

1. T is a bounded operator from $L^2(\mathbb{R}^{1+n}_+)$ to $L^2_\kappa(\mathbb{R}^{1+n}_+)$;
2. There exists $K \in \mathrm{SK}^\kappa_{m,q,M}$, called the *kernel* of T, such that the representation
$$Tf(t,y) = \int_0^t (K(t,s)f(s))(y)ds \tag{4.10}$$
holds
   - for any $f \in L^2(\mathbb{R}^{1+n}_+)$ and a.e. $(t,y) \in \mathbb{R}^{1+n}_+$, if $\kappa > 0$;
   - for any $f \in L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$ and a.e. $(t,y) \in \pi(f)^c$, if $\kappa \leq 0$.

Denote by $\mathrm{SIO}^{\kappa+}_{m,q,M}$ the set of all SIOs of type $+(\kappa, m, q, M)$.

There is a slight abuse of terminology since the operator is not singular when $\kappa > 0$.

The next class of singular integral operators is concerned with negative type. We shall present results for it since it is instrumental for duality. It can be used to obtain estimates for backward Cauchy problems, but we leave to the reader the care of stating them.

Definition 4.10 (SIO of type $-(\kappa, m, q, M)$)**.** Let $\kappa \in \mathbb{R}, m \in \mathbb{N}, 1 \leq q \leq \infty$, and $M_{\kappa,q} < M \leq \infty$ be constants. An operator T is called a *singular integral operator of type* $-(\kappa, m, q, M)$, if

1. T is a bounded operator from $L^2_{-\kappa}(\mathbb{R}^{1+n}_+)$ to $L^2(\mathbb{R}^{1+n}_+)$;
2. There exists $K \in \mathrm{SK}^\kappa_{m,q,M}$, called the *kernel* of T, such that the representation
$$Tf(t,y) = \int_t^\infty (K(t,s)f(s))(y)ds \tag{4.11}$$
holds
   - for any $f \in L^2_{-\kappa}(\mathbb{R}^{1+n}_+)$ and a.e. $(t,y) \in \mathbb{R}^{1+n}_+$, if $\kappa > 0$;
   - for any $f \in L^2_{-\kappa}(\mathbb{R}^{1+n}_+)$ with bounded support in $\overline{\mathbb{R}^{1+n}_+}$ and a.e. $(t,y) \in \pi(f)^c$, if $\kappa \leq 0$.

Denote by $\mathrm{SIO}^{\kappa-}_{m,q,M}$ the set of all SIOs of type $-(\kappa, m, q, M)$.

For $\kappa > 0$, the integrals in (4.10) and (4.11) are well-defined by Lemma 4.6. For $\kappa \leq 0$, one uses Lemma 4.8. In the case $\kappa = 0$, compared with Definition 2.1 in [AKMP12], our definitions seem more restrictive, but in fact, (4.10) and (4.11) are all what is needed in their proofs to obtain tent space estimates.

These two types of SIOs are linked via $L^2(\mathbb{R}^{1+n}_+)$-duality given by

$$\langle f, g \rangle_{L^2(\mathbb{R}^{1+n}_+)} := \int_{\mathbb{R}^{1+n}_+} f(t,y)\overline{g}(t,y)dtdy.$$

Proposition 4.11. *Let $\kappa \in \mathbb{R}, m \in \mathbb{N}, 1 \leq q \leq \infty$, and $M_{\kappa,q} < M \leq \infty$. Let T be in $\mathrm{SIO}^{\kappa\pm}_{m,q,M}$ with kernel K and T^* be the adjoint of T with respect to $L^2(\mathbb{R}^{1+n}_+)$-duality. Then T^* lies in $\mathrm{SIO}^{\kappa\mp}_{m,q',M}$ with kernel K^*.*

Proof. We only prove the case assuming $T \in \mathrm{SIO}^{\kappa+}_{m,q,M}$ with $\kappa \leq 0$. The other cases are left to the reader. The boundedness of T^* is clear, and Corollary 4.5 shows $K^* \in \mathrm{SK}^{\kappa}_{m,q',M}$, so it suffices to prove the representation. For any $f \in L^2_{-\kappa}(\mathbb{R}^{1+n}_+)$ with bounded support in $\overline{\mathbb{R}^{1+n}_+}$ and $g \in L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$ so that $\pi(g)$ is disjoint with $\pi(f)$, we get by (4.10)

$$\langle Tg, f \rangle_{L^2(\mathbb{R}^{1+n}_+)} = \int_{\mathbb{R}^{1+n}_+} \left(\int_0^s (K(s,t)g(t))(y)dt \right) \overline{f}(s,y)dsdy.$$

Thanks to Lemma 4.7, Schur test implies

$$\begin{aligned}
&\int_{\mathbb{R}^{1+n}_+} \int_0^s |(K(s,t)g(t))(y)||f(s,y)|dtdsdy \\
&= \int_0^\infty ds \int_0^s dt \int_{\mathbb{R}^n} |\mathbb{1}_{\pi(f)}(s,y)(K(s,t)g(t))(y)||f(s,y)|dy \\
&\leq \int_0^\infty ds \int_0^s \omega_{\pi(f),\pi(g)}(s,t)\|g(t)\|_2 \|f(s)\|_2 \, dt \\
&\lesssim \|g\|_{L^2(\mathbb{R}^{1+n}_+)} \|f\|_{L^2(\mathbb{R}^{1+n}_+)} \lesssim \|g\|_{L^2(\mathbb{R}^{1+n}_+)} \|f\|_{L^2_{-\kappa}(\mathbb{R}^{1+n}_+)} < \infty,
\end{aligned}$$

using that f has bounded support. Fubini's theorem ensures that

$$\begin{aligned}
\langle g, T^*f \rangle_{L^2(\mathbb{R}^{1+n}_+)} &= \langle Tg, f \rangle_{L^2(\mathbb{R}^{1+n}_+)} \\
&= \int_0^\infty ds \int_0^s \langle K(s,t)g(t), f(s) \rangle_{L^2(\mathbb{R}^n)} \, dt \\
&= \int_0^\infty dt \int_t^\infty \langle g(t), K(s,t)^* f(s) \rangle_{L^2(\mathbb{R}^n)} \, ds \\
&= \int_{\mathbb{R}^{1+n}_+} g(t,x) \left(\int_t^\infty \overline{(K(s,t)^* f(s))}(x) ds \right) dtdx.
\end{aligned}$$

We conclude by arbitrariness of g that for a.e. $(t,x) \in \pi(f)^c$,

$$T^*(f)(t,x) = \int_t^\infty (K(s,t)^* f(s))(x) ds.$$

Hence (4.11) holds for T^* with kernel K^*. $\square$

4.2 Extensions of SIOs to tent spaces

The bounded extensions of SIOs of type $\pm(0,m,q,M)$ on tent spaces have been studied in [AKMP12]. In this section, we improve their results in our framework, simplify some proofs, and generalize to SIOs of type $\pm(\kappa,m,q,M)$ for $\kappa \neq 0$.

4.2.1 Main results

In this section, we state our main results on the extension of SIOs and their adjoints to tent spaces $T^{p;m}_\beta$. This is divided into four statements depending on the type of SIOs and the range of p. Proofs are postponed to Section 4.2.2.

Throughout the section, $\kappa \in \mathbb{R}$ and $m \in \mathbb{N}$ are fixed constants.

The first statement concerns the extension of operators in $\mathrm{SIO}^{\kappa+}_{m,q,M}$ to $T^{p;m}_\beta$ for $p \leq 2$. Define

$$p_M := \frac{2n}{n+2mM}, \quad p_q(\beta) := \frac{2nq}{2n+(2\beta+1)mq}, \tag{4.12}$$

and

$$M_c(\kappa,q) := \max\left\{\frac{n}{2m}, M_{\kappa,q}\right\}, \tag{4.13}$$

where $M_{\kappa,q}$ is defined in (4.9). When $M > M_c(\kappa,q)$ then $M > M_{\kappa,q}$ so the class $\mathrm{SIO}^{\kappa+}_{m,q,M}$ is well-defined, and also $p_M < 1$.

Proposition 4.12. *Let $1 \leq q \leq 2, M_c(\kappa,q) < M \leq \infty$, and $\beta > -1/2$. Let T be in $\mathrm{SIO}^{\kappa+}_{m,q,M}$, and assume that T is a bounded operator from $L^2_\beta(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+\kappa}(\mathbb{R}^{1+n}_+)$. Then T extends to a bounded operator from $T^{p;m}_\beta$ to $T^{p;m}_{\beta+\kappa}$ when $\max\{p_M, p_q(\beta)\} < p \leq 2$, if one of the following conditions holds:*

(1) $q' > \frac{2n}{m(2\beta+1)}$ *(or equivalently $p_q(\beta) < 1$);*

(2) $2 \leq q' \leq \frac{2n}{m(2\beta+1)}$ *(or equivalently $p_q(\beta) \geq 1$) and $M > \frac{n}{mq}$.*

The next statement establishes the extension of operators in $\mathrm{SIO}^{\kappa-}_{m,q,M}$ to $T^{p;m}_\beta$ when $p \leq 2$.

Proposition 4.13. *Let $1 \leq q \leq 2, M_c(\kappa,q) < M \leq \infty$, and $\beta < 1/2$. Let T be in $\mathrm{SIO}^{\kappa-}_{m,q,M}$, and assume that T is a bounded operator from $L^2_{\beta-\kappa}(\mathbb{R}^{1+n}_+)$ to $L^2_\beta(\mathbb{R}^{1+n}_+)$. Then T extends to a bounded operator from $T^{p;m}_{\beta-\kappa}$ to $T^{p;m}_\beta$ when $p_M < p \leq 2$.*

The third statement describes the extension of operators in $\mathrm{SIO}^{\kappa+}_{m,q,M}$ on $T^{p;m}_\beta$ when $p \geq 2$.

Corollary 4.14. *Let $2 \le q \le \infty, M_c(\kappa, q) < M \le \infty$, and $\beta > -1/2$. Let T be in $\mathrm{SIO}^{\kappa+}_{m,q,M}$, and assume that T is a bounded operator from $L^2_\beta(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+\kappa}(\mathbb{R}^{1+n}_+)$. Then T extends to a bounded operator*

- *from $T^{p;m}_\beta$ to $T^{p;m}_{\beta+\kappa}$, when $2 \le p \le \infty$,*
- *from $T^{\infty;m}_{\beta,([p,1])}$ to $T^{\infty;m}_{\beta+\kappa,([p,1])}$, when $p_M < p \le 1$.*

The last statement is concerned with the extension of operators in $\mathrm{SIO}^{\kappa-}_{m,q,M}$ on $T^{p;m}_\beta$ when $p \ge 2$.

Corollary 4.15. *Let $2 \le q \le \infty, M_c(\kappa, q) < M \le \infty$, and $\beta < 1/2$. Let T be in $\mathrm{SIO}^{\kappa-}_{m,q,M}$, and assume that T is a bounded operator from $L^2_{\beta-\kappa}(\mathbb{R}^{1+n}_+)$ to $L^2_\beta(\mathbb{R}^{1+n}_+)$.*

(1) *If $q > \frac{2n}{m(2\beta+1)}$ (or equivalently $p_{q'}(\beta) < 1$), then T extends to a bounded operator*

   - *from $T^{p;m}_{\beta-\kappa}$ to $T^{p;m}_\beta$, when $2 \le p \le \infty$,*
   - *from $T^{\infty;m}_{\beta-\kappa,([p,1])}$ to $T^{\infty;m}_{\beta,([p,1])}$, when $\max\{p_M, p_{q'}(\beta)\} < p \le 1$.*

(2) *If $2 \le q \le \frac{2n}{m(2\beta+1)}$ (or equivalently $p_{q'}(\beta) \ge 1$) and $M > \frac{n}{mq'}$, then T extends to a bounded operator from $T^{p;m}_{\beta-\kappa}$ to $T^{p;m}_\beta$ when $2 \le p < (p_{q'}(\beta))'$.*

Remark 4.16. Let us compare with [AKMP12].

- We treat the new cases $\kappa \ne 0$. The constant $M_c(\kappa, q)$ depends on κ, while the exponents p_M and $p_q(\beta)$ do not.
- There is an improvement in Proposition 4.12 (2) and hence Corollary 4.15 (2) when $q \ne 2$. The range of p is larger but at the expense of the extra decay on M. This range for p was obtained in the case of a maximal regularity operator when $\kappa = 0$, see also [Hua17, Theorem 1.1]. This extra decay only comes from our argument and we do not know how to remove it. But it should be the case, as a discontinuity on the required lower control for M with respect to q appears at $q' = \frac{2n}{m(2\beta+1)}$ by looking at the two cases.

Remark 4.17. For $\kappa > 0$, the bounded extensions of $\mathrm{SIO}^{\kappa+}_{m,q,M}$-operators from $L^2_\beta(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+\kappa}(\mathbb{R}^{1+n}_+)$ are automatic, see Lemma 4.6. For $\kappa = 0$, it was asserted in [AKMP12, Theorem 2.2] but in fact the proof has a gap. It is unclear whether the operators in $\mathrm{SIO}^{0+}_{m,q,M}$ are bounded on $L^2_\beta(\mathbb{R}^{1+n}_+)$, so it needs to be assumed, and also for $\kappa < 0$. However, it is true for maximal regularity operators, see [AA11a, Theorem 1.3].

Remark 4.18. Both propositions are proved by density, while both corollaries are proved by duality arguments. See below. Usual arguments show that extension by duality agrees with extension by density in norm when $p < \infty$ and for the weak* topology when $p = \infty$.

4.2.2 Proofs

In this section, we provide the proof of results in Section 4.2.1. The arguments follow those of [AKMP12] to which we will refer, except for Proposition 4.12 (2). For this one, we still follow the setup there but introduce a new method based on extrapolation of weighted estimates to improve the range of the exponent p. We first settle the proof of the last three results.

Proof of Proposition 4.13. The proof is a verbatim adaptation of that of [AKMP12, Proposition 3.7]. We leave it to the reader. □

Proof of Corollary 4.14 and Corollary 4.15. The proofs are similar by a formal duality argument from Proposition 4.13 and Proposition 4.12, respectively. Therefore, we only prove the case $p_M < p \leq 1$ in Corollary 4.14, and the rest is left to the reader.

Proposition 4.11 says $T^* \in \mathrm{SIO}^{\kappa-}_{m,q',M}$, and by duality, T^* extends to a bounded operator from $L^2_{-\beta-\kappa}(\mathbb{R}^{1+n}_+)$ to $L^2_{-\beta}(\mathbb{R}^{1+n}_+)$. Note that $M_c(\kappa,q) = M_c(\kappa,q')$, so for any $f \in T^{\infty;m}_{\beta+\kappa,([p,1])}$, Proposition 4.13 implies that $g \mapsto \langle f, T^*g \rangle$ is a bounded anti-linear functional on $T^{p;m}_{-\beta-\kappa}$. Define $Tf \in T^{\infty;m}_{\beta,([p,1])}$ as $Tf(g) := \langle f, T^*g \rangle$ via the identification, and we get

$$\|Tf\|_{T^{\infty;m}_{\beta+\kappa,([p,1])}} = \sup_{g:\|g\|_{T^{p;m}_{-\beta-\kappa}}=1} |\langle f, T^*g \rangle| \leq \|T^*\|_{\mathscr{L}(T^{p;m}_{-\beta-\kappa}, T^{p;m}_{-\beta})} \|f\|_{T^{\infty;m}_{\beta,([p,1])}},$$

where $\mathscr{L}(X,Y)$ is the space of bounded linear operators from the (quasi-)Banach space X to the (quasi-)Banach space Y.

This completes the proof. □

We now turn to the proof of Proposition 4.12. We split the operator T into two parts, the *regular part*

$$T_0(f)(t,y) = \int_0^{t/2} (K(t,s)f(s))(y)ds,$$

and the *singular part*

$$T_{\mathrm{sing}}(f)(t,y) = \int_{t/2}^{t} (K(t,s)f(s))(y)ds.$$

The meaning of the integrals by taking the restricted singular kernels

$$\mathbb{1}_{\{s<t/2\}}(t,s)K(t,s)$$

or

$$\mathbb{1}_{\{t/2<s<t\}}(t,s)K(t,s)$$

is that of (4.3) if $\kappa > 0$ or Lemma 4.8 if $\kappa \leq 0$.

Let us start with the singular part.

Lemma 4.19. *Let T be in $\mathrm{SIO}^{\kappa+}_{m,q,M}$ with $M > M_c(\kappa,q)$ and assume that T is a bounded operator from $L^2_\beta(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+\kappa}(\mathbb{R}^{1+n}_+)$. Then, T_{sing} extends to a bounded operator from $T^{p;m}_\beta$ to $T^{p;m}_{\beta+\kappa}$ when $p_M < p \leq 2$.*

Proof. By interpolation, it suffices to consider the extension of T_{sing} to $T^{p;m}_\beta$ for $p_M < p \leq 1$. By Corollary 3.6, it suffices to prove that T_{sing} is uniformly bounded from $T^{p;m}_\beta$-atoms to $T^{p;m}_{\beta+\kappa}$. This is a verbatim adaptation of [AKMP12, Lemma 3.4] once we assume that T is bounded from $L^2_\beta(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+\kappa}(\mathbb{R}^{1+n}_+)$. We leave it to the reader. $\square$

We then move on to estimate the regular part T_0. If $q' > \frac{2n}{m(2\beta+1)}$, cf. Proposition 4.12 (1), we also obtain the uniform boundedness of T_0 on $T^{p;m}_\beta$-atoms for $p_q(\beta) < p \leq 1$ and conclude by Corollary 3.6. But it does not work if $q' \leq \frac{2n}{m(2\beta+1)}$, cf. Case (2). In this case, our strategy is to embed T_0 into the analytic family of operators $\{T_z\}_{z\in\mathbb{C}}$ given by

$$T_z(f)(t,y) := \int_0^{t/2} \left(\frac{s}{t}\right)^z (K(t,s)f(s))(y)ds. \tag{4.14}$$

If $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$, then the integral makes sense as a Bochner integral valued in $L^2(\mathbb{R}^n)$ for any $z \in \mathbb{C}$ and a.e. $(t,y) \in \mathbb{R}^{1+n}_+$.

We study the boundedness of T_0 by Stein's interpolation. We show that if $\Re(z)$ is sufficiently large, we can still obtain a uniform bound of T_z acting on atoms. The main task is to obtain boundedness when $\Re(z) < 0$. This is given by the following proposition.

Proposition 4.20. *Let K be in $\mathrm{SK}^\kappa_{m,q,M}$ with $1 \leq q < 2$ and $M > \frac{n}{mq}$. Then, for any $\beta \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\Re(z) > -\beta - \frac{1}{2}$, T_z can be extended to a bounded operator from $T^{p;m}_\beta$ to $T^{p;m}_{\beta+\kappa}$ when $q < p \leq 2$.*

Compared to [AKMP12, Lemma 3.5], this improves the range of p from $p = 2$ to $q < p \leq 2$. This is achieved via an extrapolation of weighted estimates technique. Let us start with a lemma.

Lemma 4.21. *Let $\{K(t,s)\}_{(t,s)\in\Delta^c}$ be a strongly measurable family of bounded operators on $L^2(\mathbb{R}^n)$ such that (4.2) holds for some $\kappa \in \mathbb{R}$. Then, for any $\beta \in \mathbb{R}$ and $z \in \mathbb{C}$ with $\Re(z) > -\beta - \frac{1}{2}$, the integral (4.14) converges for $f \in L^2_\beta(\mathbb{R}^{1+n}_+)$. Moreover, it defines a bounded operator from $L^2_\beta(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+\kappa}(\mathbb{R}^{1+n}_+)$.*

Proof. The proof is the same as that of (4.3) but changing the kernel $k(t,s)$ to $\mathbb{1}_{\{0<s<t/2\}}(t,s)\left(\frac{s}{t}\right)^{\Re(z)}(t-s)^{\kappa-1}s^{\beta+\frac{1}{2}}t^{-\beta-\kappa+\frac{1}{2}}$. $\square$

Remark 4.22. Lemma 4.21 says T_0 is always bounded from $L^2_\beta(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+\kappa}(\mathbb{R}^{1+n}_+)$ if $\beta > -1/2$. However, it is unclear for T_{sing} if $\kappa \leq 0$. This is why we need to assume it on T, or equivalently on T_{sing}.

Next, we bring in a pointwise estimate. For any $f \in L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$ and $x \in \mathbb{R}^n$, define the *vertical square function* as

$$\mathcal{V}_\beta(f)(x) := \left(\int_0^\infty |t^{-\beta} f(t,x)|^2 dt \right)^{1/2}.$$

Let $\mathcal{M}$ be the centred Hardy–Littlewood maximal function given by

$$\mathcal{M}(g)(x) := \sup_{\rho>0} \fint_{B(x,\rho)} |g(y)| dy$$

for any $g \in L^1_{\mathsf{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$.

Lemma 4.23. *With the assumptions of Proposition 4.20, it holds that*

$$\mathcal{A}_{\beta+\kappa;m}(T_z(f))(x) \lesssim \mathcal{V}_\beta(\mathcal{M}_q(f))(x) \tag{4.15}$$

for any $f \in L^2_\beta(\mathbb{R}^{1+n}_+)$ and $x \in \mathbb{R}^n$, where

$$\mathcal{M}_q(f)(t,x) := \sup_{\rho>0} \left(\fint_{B(x,\rho)} |f(t,y)|^q dy \right)^{1/q} = \mathcal{M}(|f(t)|^q)(x)^{1/q}.$$

Proof. From the definition (3.1), we get

$$\mathcal{A}_{\beta+\kappa;m}(T_z(f))(x) \eqsim \left\| t^{-\frac{n}{2m}-\beta-\kappa} \|T_z(f)(t)\|_{L^2(B(x,t^{1/m}))} \right\|_{L^2(\mathbb{R}_+,dt)}. \tag{4.16}$$

Fix $(t,x) \in \mathbb{R}^{1+n}_+$. Define $B_0 := B(x,t^{1/m})$, $C_0 := 2B_0$, and $C_j := 2^{j+1}B_0 \setminus 2^j B_0$ for $j \geq 1$, so that

$$\|T_z(f)(t)\|_{L^2(B_0)} \leq \sum_{j\geq 0} \|\mathbb{1}_{B_0} T_z(\mathbb{1}_{C_j} f)(t)\|_2.$$

Using $L^q - L^2$ off-diagonal decay and $t - s \sim t$, we get

$$\begin{aligned}
\|\mathbb{1}_{B_0} T_z(\mathbb{1}_{C_j} f)(t)\|_2 &\leq \int_0^{t/2} \left(\frac{s}{t}\right)^{\Re(z)} \|\mathbb{1}_{B_0} K(t,s)(\mathbb{1}_{C_j} f(s))\|_2 \, ds \\
&\lesssim 2^{-jmM} t^{-\Re(z)-1+\kappa-\frac{n}{m}[q,2]} \int_0^{t/2} s^{\Re(z)} \|\mathbb{1}_{C_j} f(s)\|_q \, ds \\
&\lesssim 2^{-j(mM-\frac{n}{q})} t^{-\Re(z)-1+\kappa+\frac{n}{2m}} \int_0^{t/2} s^{\Re(z)} \mathcal{M}_q(f(s))(x) ds,
\end{aligned}$$

due to the fact that

$$\|\mathbb{1}_{C_j} f(s)\|_q \leq \|\mathbb{1}_{2^{j+1}B_0} f(s)\|_q \lesssim_q (2^j t^{\frac{1}{m}})^{\frac{n}{q}} \mathcal{M}_q(f(s))(x).$$

Since $M > \frac{n}{mq}$, taking the sum on j, we get

$$\|T_z(f)(t)\|_{L^2(B_0)} \lesssim t^{-\Re(z)-1+\kappa+\frac{n}{2m}} \int_0^{t/2} s^{\Re(z)} \mathcal{M}_q(f(s))(x) ds.$$

By (4.16), we derive

$$\begin{aligned}\mathcal{A}_{\beta+\kappa;m}(T_z(f))(x)^2 &\lesssim \int_0^\infty \left(\int_0^{t/2} \left(\frac{s}{t}\right)^{\Re(z)+\beta+\frac{1}{2}} \mathcal{M}_q(s^{-\beta+\frac{1}{2}} f(s))(x) \frac{ds}{s} \right)^2 \frac{dt}{t} \\ &\lesssim \int_0^\infty (s^{-\beta}\mathcal{M}_q(f(s))(x))^2 ds = \mathcal{V}_\beta(\mathcal{M}_q(f))(x)^2.\end{aligned}$$

The last inequality follows by Schur test, as

$$\int_0^{t/2} \left(\frac{s}{t}\right)^{\Re(z)+\beta+\frac{1}{2}} \frac{ds}{s} \leq C(z,\beta), \quad \int_{2s}^\infty \left(\frac{s}{t}\right)^{\Re(z)+\beta+\frac{1}{2}} \frac{dt}{t} \leq C(z,\beta),$$

when $\Re(z) > -\beta - \frac{1}{2}$. □

We can now prove Proposition 4.20. Terminologies and facts related to weights w can be found in [CUMP11].

Proof of Proposition 4.20. The conclusion when $p = 2$ has been shown in Lemma 4.21. We turn to proving it for $p < 2$.

For any weight $w \in A_{2/q}$ and $f \in L^2_c(\mathbb{R}^{1+n}_+)$, Lemma 4.23 and the $L^{2/q}(w)$-boundedness of $\mathcal{M}$ imply

$$\begin{aligned}\|\mathcal{A}_{\beta+\kappa;m}(T_z(f))\|_{L^2(w)}^2 &\lesssim \|\mathcal{V}_\beta(\mathcal{M}_q(f))\|_{L^2(w)}^2 \\ &= \int_0^\infty t^{-2\beta} dt \int_{\mathbb{R}^n} \mathcal{M}(|f(t)|^q)(x)^{\frac{2}{q}} w(x) dx \\ &\lesssim \int_0^\infty t^{-2\beta} dt \int_{\mathbb{R}^n} |f(t,x)|^2 w(x) dx = \|\mathcal{V}_\beta(f)\|_{L^2(w)}^2.\end{aligned}$$

Limited-range extrapolation (see [CUMP11, Theorem 3.31]) implies for any $p \in (q,2)$, $w \in A_{p/q} \cap \mathrm{RH}_{(2/p)'}$, and $f \in L^2_c(\mathbb{R}^{1+n}_+)$,

$$\|\mathcal{A}_{\beta+\kappa;m}(T_z(f))\|_{L^p(w)} \lesssim \|\mathcal{V}_\beta(f)\|_{L^p(w)}.$$

As $p < 2$, we invoke [AHM12, Proposition 2.3(b)] and get

$$\|\mathcal{V}_\beta(f)\|_{L^p(w)} \lesssim \|\mathcal{A}_{\beta;m}(f)\|_{L^p(w)}$$

by change of scaling $t \to t^{1/m}$. Taking $w = 1$, we obtain that for any $p \in (q,2)$ and $f \in L^2_c(\mathbb{R}^{1+n}_+)$,

$$\|T_z(f)\|_{T^{p;m}_{\beta+\kappa}} \eqsim \|\mathcal{A}_{\beta+\kappa;m}(T_z(f))\|_p \lesssim \|\mathcal{A}_{\beta;m}(f)\|_p = \|f\|_{T^{p;m}_\beta}.$$

We conclude by a density argument. □

Let us finish the proof of Proposition 4.12.

Proof of Proposition 4.12. The bounded extension of T_{sing} from $T^{p;m}_\beta$ to $T^{p;m}_{\beta+\kappa}$ for $p_M < p \leq 2$ has been shown in Lemma 4.19. Consider the regular part T_0.

Case (1): $q' > \frac{2n}{m(2\beta+1)}$ We obtain the uniform $T^{p;m}_{\beta+\kappa}$-bound of T_0 on $T^{p;m}_{\beta}$-atoms when $p_q(\beta) < p \leq 1$ by adapting the calculation in [AKMP12, Lemma 3.5]. As the range of q corresponds to $p_q(\beta) < 1$, we conclude using Corollary 3.6.

Case (2): $q' \leq \frac{2n}{m(2\beta+1)}$ We first claim that T_z extends to a bounded operator from $T^{1;m}_{\beta}$ to $T^{1;m}_{\beta+\kappa}$ if $\Re(z) > \frac{n}{mq'} - \beta - \frac{1}{2} \geq 0$. Indeed, the adaptation of [AKMP12, Lemma 3.5] also implies the uniform $T^{1;m}_{\beta+\kappa}$-bound of T_z on $T^{1;m}_{\beta}$-atoms, so the claim follows by Corollary 3.6.

Then the discussion is organised into two cases.

- If $q' > 2$, Proposition 4.20 says T_z extends to a bounded operator from $T^{p;m}_{\beta}$ to $T^{p;m}_{\beta+\kappa}$ if $\Re(z) > -\beta - \frac{1}{2}$ and $q < p \leq 2$.

  Then for any $p \in (p_q(\beta), 2]$, one can find $t_0 \in (-\beta-\frac{1}{2}, 0)$, $t_1 > \frac{n}{mq'} - \beta - \frac{1}{2}$, and $q_0 \in (p, 2]$ so that

  $$(1-\theta)t_0 + \theta t_1 = 0, \quad \frac{1-\theta}{q_0} + \frac{\theta}{1} = \frac{1}{p}$$

  for some $\theta \in (0,1)$. For any $y \in \mathbb{R}$, the above discussion implies $T_{t_0+iy} \in \mathscr{L}(T^{q_0;m}_{\beta}, T^{q_0;m}_{\beta+\kappa})$ and $T_{t_1+iy} \in \mathscr{L}(T^{1;m}_{\beta}, T^{1;m}_{\beta+\kappa})$ with their operator norms independent of y. Then T_0 extends to a bounded operator from T^{p}_{β} to $T^{p}_{\beta+\kappa}$ as desired, thanks to Stein's interpolation theorem on tent spaces (see [HTV91]).

- If $q' = 2$, the adaptation of [AKMP12, Lemma 3.5] shows that T_z extends to a bounded operator from $T^{2;m}_{\beta}$ to $T^{2;m}_{\beta+\kappa}$ if $\Re(z) > -\beta - \frac{1}{2}$. We hence also get the bounded extension of T_0 from T^{p}_{β} to $T^{p}_{\beta+\kappa}$ when $p \in (p_2(\beta), 2]$ by the same argument as above.

This completes the proof. $\square$

Chapter 5

Heat equation

> "*Il ne faut donc pas croire que les théories démodées ont été stériles et vaines.*"
>
> *La Valeur de la science*,
> Henri Poincaré

In this chapter and the next two chapters, we employ the theory of tent spaces to investigate linear parabolic Cauchy problems with rough coefficients and rough initial data.

This opening chapter is devoted to the most prototypical example, the heat equation. We begin by the characterization of homogeneous Hardy–Sobolev spaces by their heat extensions in tent spaces (see Theorem 5.4 and Corollary 5.6). Then we establish a converse statement, which asserts that any distributional heat solution satisfying the tent space condition possesses a (unique) trace in the homogeneous Hardy–Sobolev space, and is represented by the heat extension of its trace (see Theorem 5.17). This result is grounded in a general representation theorem (cf. Theorem 5.7) for distributional heat solutions with interior local L^2-growth controlled by the inverse of a Gaussian (complemented by a uniform control approaching to the initial time). The last section extends these results to homogeneous Besov spaces.

Convention 5.1. Starting from this chapter, we restrict ourselves to the parabolic settings with homogeneity $m = 2$. For convenience, we write

- T^p_β for the tent space $T^{p;2}_\beta$,
- $T^\infty_{\beta,(\sigma)}$ for the tent space $T^{\infty;2}_{\beta,(\sigma)}$,
- T^∞_β for the tent space $T^{\infty;2}_{\beta,(0)}$,
- Z^p_β for the Z-space $Z^{p;2}_\beta$,

– E^p_δ for the slice space $E^{p;2}_\delta$.

We also follow the notation $[p,q] := \frac{1}{p} - \frac{1}{q}$ for any $p,q \in (0,\infty]$, if there is no confusion with closed intervals.

This chapter is a compilation of (part of) the two articles:

- "*On representation of solutions to the heat equation*" [AH24], written in collaboration with Pascal Auscher and published in *Comptes Rendus Mathématique*.

- "*On well-posedness for parabolic Cauchy problems of Lions type with rough initial data*" [AH25b], written in collaboration with Pascal Auscher and published in *Mathematische Annalen*.

5.1 Homogeneous Hardy–Sobolev spaces

In this section, we introduce the definition and some fundamental properties of the homogeneous Hardy–Sobolev spaces. Denote by $\mathscr{S}$ the space of Schwartz functions on $\mathbb{R}^n$ and by $\mathscr{S}'$ the space of tempered distributions. For any $f \in \mathscr{S}'$, write $\widehat{f}$ or $\mathcal{F}(f)$ for the Fourier transform of f.

Our homogeneous Hardy–Sobolev spaces can be roughly regarded as "realizations" of Triebel–Lizorkin spaces in $\mathscr{S}'$. We follow the formulation in [Saw18, §2.4.3 and 3.3.3], which was originated from an observation of J. Peetre [Pee76, p.56], see also [Bou13, Mou15].

Denote by C the annulus $\{\xi \in \mathbb{R}^n : 2^{-1} \leq |\xi| \leq 2^2\}$ and by 2^jC the annulus $\{\xi \in \mathbb{R}^n : 2^{j-1} \leq |\xi| \leq 2^{j+2}\}$ for any $j \in \mathbb{Z}$. Let χ be in $C^\infty_c(\mathbb{R}^n)$ so that $\operatorname{supp}(\chi) \subset C$ and for any $\xi \neq 0$,

$$\sum_{j\in\mathbb{Z}} \chi(2^{-j}\xi) = 1.$$

Let Δ_j be the j-th *Littlewood–Paley operator associated with* χ, given by

$$\Delta_j f := \mathcal{F}^{-1}\left(\chi(2^{-j}\cdot)\mathcal{F}(f)\right), \quad f \in \mathscr{S}'. \tag{5.1}$$

Suppose $s \in \mathbb{R}$ and $0 < p \leq \infty$. Let $\mathfrak{S}_{s,p}$ be the collection of sequences of measurable functions $(f_j)_{j\in\mathbb{Z}}$ on $\mathbb{R}^n$ so that

$$\|(f_j)\|_{\mathfrak{S}_{s,p}} := \left\| \left(\sum_j |2^{js} f_j|^2 \right)^{1/2} \right\|_p < \infty. \tag{5.2}$$

It is clear that $f_j \in L^p$ for any $j \in \mathbb{Z}$, whenever $(f_j) \in \mathfrak{S}_{s,p}$.

Let $\mathscr{P}$ be the space of polynomials $\mathbb{C}[x_1, \dots, x_n]$, $\mathscr{P}_0 := \{0\}$, and $\mathscr{P}_m$ be the subspace of $\mathscr{P}$ consisting of polynomials of degree less than m for $m \geq 1$. For $p \neq \infty$, the *Triebel–Lizorkin space* $\dot{F}^s_{p,2}$ consists of $f \in \mathscr{S}'/\mathscr{P}$ for which

$$\|f\|_{\dot{F}^s_{p,2}} := \|(\Delta_j f)\|_{\mathfrak{S}_{s,p}} < \infty.$$

For $p = \infty$, the space $\dot{F}^s_{\infty,2}$ consists of $f \in \mathscr{S}'/\mathscr{P}$ so that there exists $(f_j) \in \mathfrak{S}_{s,\infty}$ satisfying that $f = \sum_j \Delta_j f_j$ in $\mathscr{S}'/\mathscr{P}$. The norm is given by

$$\|f\|_{\dot{F}^s_{\infty,2}} := \inf \|(f_j)\|_{\mathfrak{S}_{s,\infty}},$$

where the infimum is taken among all $(f_j) \in \mathfrak{S}_{s,\infty}$ so that $\sum_j \Delta_j f_j = f$ in $\mathscr{S}'/\mathscr{P}$.

Define $\nu(s,p) := \max\{0, [s - \frac{n}{p}] + 1\}$. For any $f \in \dot{F}^s_{p,2}$, the Littlewood–Paley series $\sum_j \Delta_j f$ converges in $\mathscr{S}'/\mathscr{P}_{\nu(s,p)}$. Moreover, it induces an isometric embedding $\iota : \dot{F}^s_{p,2} \to \mathscr{S}'/\mathscr{P}_{\nu(s,p)}$ given by

$$\iota(f) := \sum_j \Delta_j f.$$

Definition 5.2 (Homogeneous Hardy–Sobolev spaces)**.** Let $s \in \mathbb{R}$ and $0 < p \leq \infty$. The *homogeneous Hardy–Sobolev space* $\dot{H}^{s,p}$ consists of $f \in \mathscr{S}'$ whose class in $\mathscr{S}'/\mathscr{P}_{\nu(s,p)}$ belongs to $\iota(\dot{F}^s_{p,2})$. The (quasi-)semi-norm of $\dot{H}^{s,p}$ is given by

$$\|f\|_{\dot{H}^{s,p}} := \|[f]\|_{\dot{F}^s_{p,2}},$$

where $[f]$ denotes the class of f in $\mathscr{S}'/\mathscr{P}$.

For $s = 0$, up to equivalent (quasi-)norms, $\dot{H}^{0,p}$ identifies with the Hardy space H^p if $p \leq 1$, the Lebesgue space L^p if $1 < p < \infty$, and BMO if $p = \infty$. It follows by classical results of Littlewood–Paley theory, see *e.g.*, [Tri83, §5.2.4].

We list some fundamental properties of the homogeneous Hardy–Sobolev spaces, which are straightforward analogues of those in Triebel–Lizorkin spaces, induced by the isomorphism ι.

(i) (Density) Denote by $\mathscr{S}_\infty$ the subspace of $\mathscr{S}$ consisting of $\phi \in \mathscr{S}$ so that for any multi-index α, $\partial^\alpha \widehat{\phi}(0) = 0$. It is a dense subspace of $\dot{H}^{s,p}$ if $p < \infty$, or weak*-dense if $p = \infty$.

(ii) (Duality) Let $s \in \mathbb{R}$ and $1 \leq p < \infty$. The dual of $\dot{H}^{s,p}$ identifies with $\dot{H}^{-s,p'}$ via $L^2(\mathbb{R}^n)$-duality. Particularly, for $s = 0$ and $p = 1$, it corresponds to the renowned $H^1 - \mathrm{BMO}$ duality.

(iii) (Complex interpolation) Let $s_0, s_1 \in \mathbb{R}$, and $0 < p_0 < p_1 \leq \infty$. Suppose there exists $\theta \in (0,1)$ so that

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then the complex interpolation space $[\dot{H}^{s_0,p_0}, \dot{H}^{s_1,p_1}]_\theta$ identifies with $\dot{H}^{s,p}$.

(iv) (Lifting property) Let $s, \sigma \in \mathbb{R}$ and $0 < p \leq \infty$. The *Riesz potential*

$$I_\sigma(f) := \mathcal{F}^{-1}\left(|\cdot|^{-\sigma}\mathcal{F}(f)\right)$$

is an isomorphism from $\dot{H}^{s,p}/\mathscr{P}_{\nu(s,p)}$ to $\dot{H}^{s+\sigma,p}/\mathscr{P}_{\nu(s+\sigma,p)}$. Thus, for any $s \in \mathbb{R}$, $\dot{H}^{s,\infty}/\mathscr{P}_{\nu(s,\infty)}$ is isomorphic to BMO^s introduced by Strichartz [Str80].

(v) (Sobolev embedding) Let $0 < p_0 < p_1 < \infty$ and $s_0 > s_1$ with

$$s_0 - \frac{n}{p_0} = s_1 - \frac{n}{p_1}.$$

Then $\dot{H}^{s_0,p_0}$ embeds into $\dot{H}^{s_1,p_1}$.

5.2 Heat extension

For any $f \in \mathscr{S}'$, the function

$$\mathcal{E}_{-\Delta}(f)(t,x) := (e^{t\Delta}f)(x) \tag{5.3}$$

belongs to $C^\infty(\mathbb{R}^{1+n}_+) \cap C([0,\infty); \mathscr{S}')$. We call it the *heat extension* of f.

Proposition 5.3 (Heat extension on $\dot{H}^{s,p}$)**.** *Let $s \in \mathbb{R}$ and $0 < p \leq \infty$. Then $\mathcal{E}_{-\Delta}$ is a bounded and continuous map from $\dot{H}^{s,p}$ to $C_0([0,\infty); \dot{H}^{s,p}) \cap C^\infty((0,\infty); \dot{H}^{s,p})$ with the estimate*

$$\sup_{t \geq 0} \|\mathcal{E}_{-\Delta}(f)(t)\|_{\dot{H}^{s,p}} \eqsim \|f\|_{\dot{H}^{s,p}}. \tag{5.4}$$

Here, for $p = \infty$, the space $\dot{H}^{s,\infty}$ is equipped with the weak-topology against elements of $\dot{H}^{-s,1}$.*

Proof. First consider the case $0 < p < \infty$. Let (Δ_j) be the Littlewood–Paley operator defined in (5.1). It is well-known that (see *e.g.*, [Tri83, §5.2.2]) for any $s \in \mathbb{R}$ and $f \in \mathscr{S}_\infty$,

$$\sup_{t \geq 0} \left\| \left(\sum_{j \in \mathbb{Z}} |2^{js}\Delta_j(e^{t\Delta}f)|^2 \right)^{1/2} \right\|_p \lesssim \left\| \left(\sum_{j \in \mathbb{Z}} |2^{js}\Delta_j f|^2 \right)^{1/2} \right\|_p .$$

Here, the implicit constant is independent of f. Thus, by density, this inequality extends to all $\dot{H}^{s,p}$, so we have

$$\sup_{t\geq 0} \|\mathcal{E}_{-\Delta}(f)(t)\|_{\dot{H}^{s,p}} \lesssim \|f\|_{\dot{H}^{s,p}}.$$

Moreover, one can find that $\mathcal{E}_{-\Delta}$ is a bounded map from $\mathscr{S}_\infty$ to $C_0([0,\infty);\mathscr{S}_\infty)\cap C^\infty((0,\infty);\mathscr{S}_\infty)$. Hence, by density, it is bounded from $\dot{H}^{s,p}$ to $C_0([0,\infty);\dot{H}^{s,p})\cap C^\infty((0,\infty);\dot{H}^{s,p})$.

For $p=\infty$, we proceed by weak*-duality to obtain boundedness and regularity. This completes the proof. $\square$

We now provide an intermediate result describing when the heat extension of a tempered distribution belongs to a weighted tent space in terms of its Hardy–Sobolev regularity. To the best of our knowledge, such a precise result and the next ones below do not appear in the literature.

Theorem 5.4 (Characterization of $\dot{H}^{s,p}$ via heat extension)**.** *Let $s\in\mathbb{R}$, $0<p\leq\infty$, and $f\in\mathscr{S}'$.*

(i) *Suppose $s<0$. If $f\in\dot{H}^{s,p}$, then $\mathcal{E}_{-\Delta}(f)$ belongs to $T^p_{(s+1)/2}$ with the estimate*

$$\|\mathcal{E}_{-\Delta}(f)\|_{T^p_{(s+1)/2}} \lesssim \|f\|_{\dot{H}^{s,p}}. \tag{5.5}$$

(ii) *Suppose $\mathcal{E}_{-\Delta}(f)$ belongs to $T^p_{(s+1)/2}$.*

(1) *If $s\geq 0$ and $\frac{n}{n+s}\leq p\leq\infty$, then $f=0$.*

(2) *If $s<0$, then f belongs to $\dot{H}^{s,p}$ with the estimate*

$$\|f\|_{\dot{H}^{s,p}} \lesssim \|\mathcal{E}_{-\Delta}(f)\|_{T^p_{(s+1)/2}}.$$

Consequently, when $s<0$, a tempered distribution $f\in\mathscr{S}'$ lies in $\dot{H}^{s,p}$ if and only if $\mathcal{E}_{-\Delta}(f)$ lies in $T^p_{(s+1)/2}$, with the equivalence of norms as

$$\|f\|_{\dot{H}^{s,p}} \eqsim \|\mathcal{E}_{-\Delta}(f)\|_{T^p_{(s+1)/2}}.$$

Remark 5.5. The restriction on s in (i) is as in Triebel–Lizorkin theory. For (ii), whenever $s\in\mathbb{R}$, $\mathcal{E}_{-\Delta}(f)\in T^p_{(s+1)/2}$ implies that there is a representative in the class of f in $\mathscr{S}'/\mathscr{P}$ that belongs to $\dot{H}^{s,p}$. But for $s\geq 0$ and $p<\frac{n}{n+s}$, we do not know any better conclusion.

In the sequel, for a function $F(t,x)$, we also write $\|F(t)\|_{T^p_\beta}$ for $\|F\|_{T^p_\beta}$ to emphasize the time variable.

Proof of Theorem 5.4. Let us start from some general facts on relations of Hardy spaces and tent spaces. Let ψ be a non-zero smooth function in $C^\infty(\mathbb{R}^n)$ satisfying that

(a) ψ is of mean zero, *i.e.*, $\int_{\mathbb{R}^n} \psi = 0$;

(b) there exists some $\varepsilon > 0$ so that for any multi-index α,

$$|\partial_x^\alpha \psi(x)| \lesssim (1+|x|)^{-n-\varepsilon-|\alpha|}. \tag{5.6}$$

Set $\psi_t(x) := t^{-n/2}\psi(t^{-1/2}x)$. One can deduce from classical arguments that the map Q_ψ defined by

$$Q_\psi(f)(t,x) := (\psi_t * f)(x)$$

is bounded from $\dot{H}^{0,p}$ to $T^p_{1/2}$, for $0 < p \leq \infty$, or equivalently, from H^p to $T^p_{1/2}$ when $0 < p \leq 1$, from L^p to $T^p_{1/2}$ when $1 < p < \infty$, and from BMO to $T^\infty_{1/2}$.

Indeed, for $p = 2$, it is direct from Fourier transform as

$$\|Q_\psi f\|^2_{T^2_{1/2}} \eqsim \|Q_\psi f\|^2_{L^2_{1/2}(\mathbb{R}^{1+n}_+)} \eqsim \int_{\mathbb{R}^n} \left(\int_0^\infty |\widehat{\psi}(t^{1/2}\xi)|^2 \frac{dt}{t} \right) |\widehat{f}(\xi)|^2 d\xi \eqsim \|f\|_2^2.$$

For $p = \infty$, it holds by a well-known Carleson measure argument, see for instance, [Gra14, Theorem 3.3.8(c)]. For $0 < p \leq 1$, it follows from atomic decomposition of tent spaces (see [CMS85, Proposition 5]), using decay and regularity of the kernel $t^{-1/2}\psi(t^{-1/2}(x-y))$ given by (5.6) and the moment conditions of H^p-atoms. Details are left to the reader. The rest follows by interpolation.

Moreover, let $\phi \in \mathscr{S}_\infty$. For $0 < p < \infty$, we infer from [CMS85, Theorem 6] (adapted to the parabolic scaling) that for any $F \in T^p_{1/2}$, the integral

$$S_\phi(F) := \int_0^\infty \phi_t * F(t) \, \frac{dt}{t}$$

converges in $\mathscr{S}'$, and hence induces a bounded operator from $T^p_{1/2}$ to L^p if $1 < p < \infty$ and to H^p if $0 < p \leq 1$. For $p = \infty$, $S_\phi(F)$ is defined in BMO by testing against $h \in H^1$ using

$$\langle S_\phi(F), h \rangle := \iint_{\mathbb{R}^{1+n}_+} F(t,x)\, \overline{Q_{\widetilde{\phi}} h(t,x)} \, \frac{dtdx}{t},$$

where $\widetilde{\phi}(x) := \overline{\phi(-x)}$. Since $\widetilde{\phi}$ also satisfies the conditions (a) and (b), by duality, one can easily get S_ϕ is bounded from $T^\infty_{1/2}$ to BMO.

Then we proceed to prove (i), assuming $s < 0$. For $0 < p < \infty$, we use density. Pick $f \in \mathscr{S}_\infty$ and define $g := (-\Delta)^{s/2} f$, so we have for any $t > 0$,

$$\mathcal{E}_{-\Delta}(f)(t) = e^{t\Delta} f = (-\Delta)^{-s/2} e^{t\Delta} g = t^{s/2}(-t\Delta)^{-s/2} e^{t\Delta} g = t^{s/2}\psi_t * g,$$

where ψ denotes the kernel of $(-\Delta)^{-s/2}e^{\Delta}$. One can easily verify that ψ satisfies the conditions (a) and (b) with $\varepsilon = -s > 0$. Then we get

$$\|\mathcal{E}_{-\Delta}(f)\|_{T^p_{(s+1)/2}} = \|Q_\psi(g)\|_{T^p_{1/2}} \lesssim \|g\|_{\dot{H}^{0,p}} = \|(-\Delta)^{s/2} f\|_{\dot{H}^{0,p}} \eqsim \|f\|_{\dot{H}^{s,p}}.$$

Since $\mathscr{S}_\infty$ is dense in $\dot{H}^{s,p}$, this inequality extends to all $f \in \dot{H}^{s,p}$.

For $p = \infty$, we use weak*-density. For any $f \in \dot{H}^{s,\infty}$, pick (f_k) as a sequence in $\mathscr{S}_\infty$ that converges weakly* to f in $\dot{H}^{s,\infty}$, hence in $\mathscr{S}'$, since $\dot{H}^{s,\infty}$ embeds into $\mathscr{S}'$ when $s < 0$. Using the above computation, we have

$$\|\mathcal{E}_{-\Delta}(f_k)\|_{T^\infty_{(s+1)/2}} \lesssim \|f_k\|_{\dot{H}^{s,\infty}}.$$

In particular, $(\mathcal{E}_{-\Delta}(f_k))$ is bounded in $T^\infty_{(s+1)/2}$. Let F be a weak*-accumulation point of $(\mathcal{E}_{-\Delta}(f_k))$ in $T^\infty_{(s+1)/2}$. Let $G \in C^\infty_c(\mathbb{R}^{1+n}_+)$. Then

$$\iint_{\mathbb{R}^{1+n}_+} (e^{t\Delta} f_k)(x)\overline{G}(t,x)\, dtdx = \int_0^\infty \langle f_k, e^{t\Delta} G(t)\rangle\, dt,$$

where the pairing is in the sense of tempered distributions and Schwartz functions. As $t \mapsto e^{t\Delta}G(t) \in C((0,\infty);\mathscr{S})$ and the integral is supported on a compact subset of $(0,\infty)$, we obtain at the limit

$$\iint_{\mathbb{R}^{1+n}_+} F(t,x)\overline{G}(t,x)\, dtdx = \int_0^\infty \langle f, e^{t\Delta} G(t)\rangle\, dt.$$

So $F = \mathcal{E}_{-\Delta}(f)$ and $\|\mathcal{E}_{-\Delta}(f)\|_{T^\infty_{(s+1)/2}} \lesssim \|f\|_{\dot{H}^{s,\infty}}$. This finishes the proof of (i).

Next, we prove (ii) and begin with (2). Let $s < 0$ and $f \in \mathscr{S}'$ with $F(t,x) := (t^{-s/2}e^{t\Delta}f)(x) \in T^p_{1/2}$. Pick $\phi \in \mathscr{S}_\infty$ with the non-degeneracy condition

$$\int_0^\infty \widehat{\phi}(r\xi)|r\xi|^{-s}e^{-|r\xi|^2}\frac{dr}{r} = 1, \quad \forall \xi \neq 0. \tag{5.7}$$

We know from the above discussion that $g := S_\phi(F)$ lies in $\dot{H}^{0,p}$. Define $\psi := (-\Delta)^{-s/2}\phi$. Computing in $\mathscr{S}'/\mathscr{P}$ and using (5.7), we have

$$(-\Delta)^{-s/2} g = S_\psi(e^{t\Delta}f) = f \quad \text{ in } \mathscr{S}'/\mathscr{P}.$$

Thus, there is a representative $\widetilde{f} \in \mathscr{S}'$ in the class of f in $\mathscr{S}'/\mathscr{P}$ which also lies in $\dot{H}^{s,p}$. By (i), we know that $e^{t\Delta}\widetilde{f}$ lies in $T^p_{(s+1)/2}$, thus, so does $e^{t\Delta}(\widetilde{f} - f)$. As $\widetilde{f} - f$ is a polynomial, $e^{t\Delta}(\widetilde{f} - f)(x) = P(t,x)$ is also a polynomial, and it is easy to see that $P(t,x) = 0$ from $\|P\|_{T^p_{(s+1)/2}} < \infty$. Finally, as $e^{t\Delta}(\widetilde{f}-f) \to \widetilde{f}-f$ in $\mathscr{S}'$ when $t \to 0$, it follows that $\widetilde{f} = f$. This proves (2).

To establish (1), we divide the discussion into two cases.

Case 1: $(s,p) \neq (0,\infty)$ For $0 < a < 1$, define

$$I(a) := \fint_a^{2a} \langle e^{t\Delta}f, \phi\rangle dt.$$

We claim that $I(a)$ tends to 0 as $a \to 0$. Meanwhile, $\langle e^{t\Delta} f, \phi \rangle$ tends to $\langle f, \phi \rangle$ as $t \to 0$, so we get $f = 0$. Let us verify the claim. For $N \geq 0$, denote by $\mathcal{P}_N$ the norm on $\mathscr{S}$ defined by

$$\mathcal{P}_N(\phi) := \sup_{|\alpha|+|\gamma| \leq N} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\gamma \phi(x)|. \tag{5.8}$$

For $1 < p \leq \infty$, pick $N > n/p'$ so that

$$\begin{aligned} \|\mathbb{1}_{(a,2a)} \phi\|_{T^{p'}_{-(s+1)/2}} &\lesssim a^{\frac{s}{2}+1} \left(\int_{\mathbb{R}^n} \langle x \rangle^{-Np'} \Big(\sup_{y \in B(x,(2a)^{1/2})} \langle y \rangle^N |\phi(y)| \Big)^{p'} dx \right)^{1/p'} \\ &\lesssim a^{\frac{s}{2}+1} \mathcal{P}_N(\phi), \end{aligned}$$

where $\langle x \rangle$ is the Japanese bracket, *i.e.*, $\langle x \rangle := (1 + |x|^2)^{1/2}$. We get

$$\begin{aligned} I(a) &\lesssim a^{-1} \|\mathbb{1}_{(a,2a)} e^{t\Delta} f\|_{T^p_{(s+1)/2}} \|\mathbb{1}_{(a,2a)} \phi\|_{T^{p'}_{-(s+1)/2}} \\ &\lesssim a^{\frac{s}{2}} \mathcal{P}_N(\phi) \|\mathbb{1}_{(a,2a)} e^{t\Delta} f\|_{T^p_{(s+1)/2}}. \end{aligned}$$

When $s > 0$, it follows that $I(a)$ tends to 0 if $a \to 0$ as $\|\mathbb{1}_{(a,2a)} e^{t\Delta} f\|_{T^p_{(s+1)/2}} \leq \|e^{t\Delta} f\|_{T^p_{(s+1)/2}}$. When $s = 0$ but $p \neq \infty$, it also holds as $\|\mathbb{1}_{(a,2a)} e^{t\Delta} f\|_{T^p_{(s+1)/2}}$ tends to 0 if $a \to 0$.

For $\frac{n}{n+s} \leq p \leq 1$, *i.e.*, $s - n[p,1] \geq 0$, note that

$$\|\mathbb{1}_{(a,2a)} \phi\|_{T^\infty_{-((s+1)/2),([p,1])}} \lesssim a \mathcal{P}_0(\phi),$$

so the claim follows as

$$I(a) \lesssim a^{-1} \|\mathbb{1}_{(a,2a)} e^{t\Delta} f\|_{T^p_{(s+1)/2}} \|\mathbb{1}_{(a,2a)} \phi\|_{T^\infty_{-((s+1)/2),([p,1])}} \lesssim_\phi \|\mathbb{1}_{(a,2a)} e^{t\Delta} f\|_{T^p_{(s+1)/2}},$$

which also tends to 0 when $a \to 0$.

Case 2: $s = 0$ and $p = \infty$ For any $\phi \in C^\infty_c(\mathbb{R}^n)$, pick a ball $B \subset \mathbb{R}^n$ containing $\operatorname{supp}(\phi)$. Note that

$$\int_0^{r(B)^2} |\langle e^{t\Delta} f, \phi \rangle|^2 \frac{dt}{t} \lesssim \|e^{t\Delta} f\|^2_{T^\infty_{1/2}} \mathcal{P}_0(\phi)^2 |B|^2 < \infty.$$

The fact that $\langle e^{t\Delta} f, \phi \rangle$ tends to $\langle f, \phi \rangle$ if $t \to 0$ forces $f = 0$ for the integral to converge, so $f = 0$ in $\mathscr{D}'(\mathbb{R}^n)$. This completes the proof. □

Replacing the heat extension by its gradient gives us the following

Corollary 5.6 (Characterization of $\dot{H}^{s,p}$ via gradient of heat extension)**.** *Let $s \in \mathbb{R}$ and $0 < p \leq \infty$.*

(i) *Suppose* $s < 1$ *and* $f \in \dot{H}^{s,p}$. *Then* $\nabla \mathcal{E}_{-\Delta}(f)$ *lies in* $T^p_{s/2}$ *with*

$$\|\nabla \mathcal{E}_{-\Delta}(f)\|_{T^p_{s/2}} \lesssim \|f\|_{\dot{H}^{s,p}}.$$

(ii) *Suppose* $f \in \mathscr{S}'$ *with* $\nabla \mathcal{E}_{-\Delta}(f) \in T^p_{s/2}$.

(1) *If* $s \geq 1$ *and* $\frac{n}{n+s-1} \leq p \leq \infty$, *then* f *is a constant.*

(2) *If* $s < 1$, *then there exists some constant* $c \in \mathbb{C}$ *so that* $f - c \in \dot{H}^{s,p}$ *with*

$$\|f - c\|_{\dot{H}^{s,p}} \eqsim \|\nabla \mathcal{E}_{-\Delta}(f)\|_{T^p_{s/2}}.$$

Consequently, when $s < 1$, *a tempered distribution* $f \in \mathscr{S}'$ *belongs to* $\dot{H}^{s,p} + \mathbb{C}$ *if and only if* $\nabla \mathcal{E}_{-\Delta}(f)$ *lies in* $T^p_{s/2}$, *with the equivalence of norms as*

$$\inf_{c \in \mathbb{C}} \|f - c\|_{\dot{H}^{s,p}} \eqsim \|\nabla \mathcal{E}_{-\Delta}(f)\|_{T^p_{s/2}}.$$

Proof. A tempered distribution $f \in \mathscr{S}'$ lies in $\dot{H}^{s,p} + \mathbb{C}$ if and only if $\nabla f \in \dot{H}^{s-1,p}$, with the equivalence $\inf_{c \in \mathbb{C}} \|f - c\|_{\dot{H}^{s,p}} \eqsim \|\nabla f\|_{\dot{H}^{s-1,p}}$. So the corollary follows from applying Theorem 5.4 to ∇f instead of f and using the above equivalence. $\square$

In fact, working modulo constants, one can equip $\dot{H}^{s,p} + \mathbb{C}$ with the "norm modulo constants" given by $\inf_{c \in \mathbb{C}} \|f - c\|_{\dot{H}^{s,p}}$. When $\nu(s,p) \geq 0$, constants are contained in $\dot{H}^{s,p}$, so $\dot{H}^{s,p} + \mathbb{C} = \dot{H}^{s,p}$, and for any $c \in \mathbb{C}$, $\|f - c\|_{\dot{H}^{s,p}} = \|f\|_{\dot{H}^{s,p}}$.

5.3 Representation of heat solutions

The purpose of this section is to investigate representation for solutions to the heat equation

$$\partial_t u - \Delta u = 0 \tag{5.9}$$

on the upper-half space $\mathbb{R}^{1+n}_+ := (0, \infty) \times \mathbb{R}^n$ or on a strip $(0, T) \times \mathbb{R}^n$. That is, when can we assert that u can be represented by the heat semigroup acting on a data, *i.e.*,

$$u(t) := u(t, \cdot) = e^{t\Delta} u_0 \tag{5.10}$$

for some u_0 and all $t \in (0, T)$?

The topic is not new, of course, so let us first briefly comment on some classical results in the literature.

The most general framework for such a representation is via tempered distributions. More precisely, given $u_0 \in \mathscr{S}'(\mathbb{R}^n)$, then $t \mapsto e^{t\Delta} u_0$ lies in $C^\infty([0, \infty); \mathscr{S}'(\mathbb{R}^n))$. Conversely, it has been shown in [Tay11, Chapter 3, Proposition 5.1] that any $u \in C^\infty([0, \infty); \mathscr{S}'(\mathbb{R}^n))$ solving the heat equation is represented by the heat semigroup applied to its initial value. Certainly, the argument still works in $C^1((0,\infty);\mathscr{S}'(\mathbb{R}^n))\cap C([0,\infty);\mathscr{S}'(\mathbb{R}^n))$, which seems to close the topic. But it uses Fourier transform, so it is not transposable to more general equations (*e.g.*, parabolic equations with coefficients). Thus, one may wonder whether different concrete knowledge, like a growth condition, on the solution could lead to a representation, not using Fourier transform. Yet, one can observe that growth exceeding the inverse of a Gaussian when $|x|\to\infty$ is forbidden for the representation.

Another framework is that of non-negative solutions. A classical result by D. Widder [Wid44, Theorem 6] shows that in one-dimensional case, any non-negative C^2-solution u in the strip must be of the form (5.10) for some non-negative Borel measure u_0. It has been generalized to higher dimensions and classical solutions of parabolic equations with smooth coefficients by M. Krzyzanski [Krz64], via internal representation and a limiting argument. We are also going to use this idea below, but we want to remove the sign condition. D. G. Aronson later extended it to non-negative weak solutions of real parabolic equations, see [Aro68, Theorem 11].

Next, the uniqueness problem is tied with representation but they are different issues. For instance, let us mention the pioneering work on non-uniqueness by A. Tychonoff [Tyc35], and two works giving sufficient criteria on strips for uniqueness. One by S. Täcklind [Täc36] provides the optimal pointwise growth condition, and the other by A. Gushchin [Gus82] provides a local L^2 condition with prescribed growth, also optimal but more amenable to more general equations. In these results, the growth can be faster than the inverse of a Gaussian when $|x|\to\infty$, which hence excludes usage of tempered distributions, so uniqueness can hold without being able to represent general solutions.

With these observations in mind, it seems that we have two very different theories to approach representation (and uniqueness): one only using distributions and Fourier transform; one not using them at all. The goal of this section is to make a bridge between them, *i.e.*, to obtain tempered distributions, not just measurable functions or measures, as initial data from local integrability conditions. Such conditions may only include integrability conditions in the interior, completed by a uniform control.

5.3.1 Main results

We say a sequence of tempered distributions (T_k) is *bounded* if $(\langle T_k,\varphi\rangle)$ is bounded for any $\varphi\in\mathscr{S}(\mathbb{R}^n)$. Recall that any distributional solution to the heat equation on strips is in fact smooth by hypoellipticity, see for instance [Hör03, § 4.4].

Theorem 5.7 (Representation of heat solutions)**.** *Let* $0<T\le\infty$*. Let* $u\in\mathscr{D}'((0,T)\times\mathbb{R}^n)$ *be a distributional solution to the heat equation. Suppose*

that:

(i) (Size condition) *For $0 < a < b < T$, there exist $C(a,b) > 0$ and $0 < \gamma < 1/4$ such that for any $R > 0$,*

$$\left(\int_a^b \int_{B(0,R)} |u(t,x)|^2 dt dx\right)^{1/2} \leq C(a,b) \exp\left(\frac{\gamma R^2}{b-a}\right); \tag{5.11}$$

(ii) (Uniform control) *There exists a sequence (t_k) tending to 0 such that $(u(t_k))$ is bounded in $\mathscr{S}'(\mathbb{R}^n)$.*

Then there exists a unique $u_0 \in \mathscr{S}'(\mathbb{R}^n)$ so that $u(t) = e^{t\Delta}u_0$ for all $0 < t < T$, where the heat semigroup is understood in the sense of tempered distributions.

Remark 5.8. The L^2 condition (i) is only assumed on interior strips, and its growth is in the order of the inverse of a Gaussian. It can be proved that this condition alone implies that $u(t)$ is a tempered distribution for each $t > 0$. The precise behavior of $C(a,b)$ is not required, and it could blow up as $a \to 0$. Condition (ii) provides us with necessary uniformity to define the initial data, which *a priori* does not follow from (i).

Remark 5.9. The conclusion also shows that u has a natural extension to a solution of the heat equation on $(0,\infty) \times \mathbb{R}^n$ if $T < \infty$. And if T was already ∞, then we could get a control on u when $t \to \infty$ via any knowledge we might get on u_0, *e.g.*, if $u_0 \in L^2(\mathbb{R}^n)$ then $u(t)$ is bounded in $L^2(\mathbb{R}^n)$.

As an interesting remark, the argument is not using Fourier transform at all, and it indeed extends with an analogous strategy to more general parabolic equations with bounded measurable, real or complex coefficients at the expense of working in more restrictive spaces than $\mathscr{S}'(\mathbb{R}^n)$, as we shall see in Section 5.4. The reader can refer to [AMP19] and [Zat20] for the proof and applications of the general result in the context of measurable initial data.

The consequence for uniqueness is as follows.

Corollary 5.10. *Let $0 < T \leq \infty$. Let $u \in \mathscr{D}'((0,T) \times \mathbb{R}^n)$ be a distributional solution to the heat equation. Suppose that (i) holds and that there exists a sequence (t_k) tending to 0 such that $(u(t_k))$ converges to 0 in $\mathscr{S}'(\mathbb{R}^n)$. Then $u = 0$.*

Remark 5.11. Convergence in $\mathscr{S}'(\mathbb{R}^n)$ cannot be replaced by convergence in $\mathscr{D}'(\mathbb{R}^n)$. The reader can refer to [CK94] for a non-identically zero solution $u \in C^\infty(\mathbb{R}^2_+) \cap C([0,\infty) \times \mathbb{R})$ with $u(0,x) = 0$ everywhere and $|u(t,x)| \leq C(\epsilon)e^{\epsilon/t}$ for any $\epsilon > 0$. The continuity implies uniform convergence of $u(t)$ to 0 on compact intervals as $t \to 0$, and hence convergence in distributional sense.

5.3.2 Proof of Theorem 5.7

Our main lemma asserts that the L^2-growth on rectangles of caloric function implies internal semigroup representation, which is an interesting fact in its own sake that we did not find in the literature. It can be seen as a particular case of [AMP19, Theorem 5.1] obtained for general parabolic equations with time-dependent, bounded measurable and elliptic coefficients. For the heat equation, an elementary proof based on Green's formula and basic estimates will be given.

Lemma 5.12 (Homotopy identity). *Let $u \in \mathscr{D}'((a,b)\times\mathbb{R}^n)$ be a distributional solution to the heat equation. Suppose that (5.11) holds. Then when $a < s < t < b$, for any $h \in C_c^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} u(t,x)h(x)dx = \int_{\mathbb{R}^n} u(s,x)(e^{(t-s)\Delta}h)(x)dx \tag{5.12}$$

where the right-hand integral absolutely converges.

The identity (5.12) can heuristically be called "*homotopy identity*", as it formally shows $u(t) = e^{(t-s)\Delta}u(s)$ in the sense of distributions. In fact, once (5.12) is shown, one can extend it to all $h \in \mathscr{S}(\mathbb{R}^n)$, so that this holds in the sense of tempered distributions.

Proof. First observe that u in fact lies in $C^\infty((a,b)\times\mathbb{R}^n)$ by hypoellipticity, as we pointed out before Theorem 5.7. Thus, all integrals and integration by parts below are justified.

Then for fixed $h \in C_c^\infty(\mathbb{R}^n)$ and $t \in \mathbb{R}$, define

$$\varphi(\tau,x) := (e^{(t-\tau)\Delta}h)(x),$$

which satisfies $\partial_\tau\varphi + \Delta\varphi = 0$ on $(-\infty,t)\times\mathbb{R}^n$. Green's formula implies for any $r > 0$ and $\tau \in (0,t)$,

$$\int_{B(0,r)} (u\Delta\varphi - \varphi\Delta u)(\tau) = \int_{\partial B(0,r)} (u\nabla\varphi - \varphi\nabla u)(\tau)\cdot\mathbf{n}\,d\sigma, \tag{5.13}$$

where $\mathbf{n}$ is the outer unit normal vector and $d\sigma$ is the sphere volume form. Here and in the sequel, unspecified measures are Lebesgue measures. Newton-Leibniz formula yields

$$\int_s^t\int_{B(0,r)} (\varphi\partial_\tau u + u\partial_\tau\varphi) = \int_{B(0,r)} (u(t)\varphi(t) - u(s)\varphi(s)). \tag{5.14}$$

Integrating (5.13) over $s < \tau < t$ and adding it to (5.14), we have

$$\int_{B(0,r)} u(t)\varphi(t) - \int_{B(0,r)} u(s)\varphi(s) = \int_s^t\int_{\partial B(0,r)} (\varphi\nabla u - u\nabla\varphi)\cdot\mathbf{n}\,d\sigma d\tau. \tag{5.15}$$

Note that $u(t)\varphi(t) \in L^1(\mathbb{R}^n)$ as $\varphi(t) = h$ has compact support. We claim $u(s)\varphi(s)$ also lies in $L^1(\mathbb{R}^n)$. Indeed, pick $\rho > 0$ such that $\operatorname{supp}(h) \subset B(0,\rho)$. Let $\kappa > 1$ be a constant to be determined. Denote by C_0 the ball $B(0,\kappa\rho)$ and by C_j the annulus $\{x \in \mathbb{R}^n : \kappa^j\rho \leq |x| < \kappa^{j+1}\rho\}$ for $j \geq 1$. We have

$$\int_{\mathbb{R}^n} |u(s)||e^{(t-s)\Delta}h| \leq \sum_{j\geq 0} \|u(s)\|_{L^2(C_j)}\|e^{(t-s)\Delta}h\|_{L^2(C_j)}. \tag{5.16}$$

Note that only terms with $j \gg 1$ are of concern as both $u(s)$ and $\varphi(s)$ are bounded on compact sets. Caccioppoli's inequality[1] and (i) imply

$$\begin{aligned}\|u(s)\|^2_{L^2(C_j)} &\lesssim_\kappa \left(\frac{1}{(\kappa^{j+1}\rho)^2} + \frac{1}{s-a}\right)\int_a^b \|u(\tau)\|^2_{L^2(B(x,\kappa^{j+2}\rho))}d\tau \\ &\lesssim_{\rho,a,b} \frac{1}{s-a}\exp\left(\frac{2\gamma}{b-a}(\kappa^{j+2}\rho)^2\right).\end{aligned}$$

The heat kernel representation implies

$$\|e^{(t-s)\Delta}h\|_{L^2(C_j)} \lesssim \exp\left(-\frac{cd_j^2}{t-s}\right)\|h\|_2$$

for $0 < c < 1/4$ and $d_j := \operatorname{dist}(C_j, \operatorname{supp}(h))$, which asymptotically equals to $\kappa^j\rho$. Pick c close to $1/4$ and κ close to 1 so that $\gamma < c/\kappa^4$, and thus for $j \gg 1$,

$$\gamma < \frac{cd_j^2}{(\kappa^{j+2}\rho)^2} < \frac{cd_j^2}{(\kappa^{j+2}\rho)^2}\cdot\frac{b-a}{t-s}.$$

It ensures that the sum in (5.16) converges and the claim hence follows.

Thus, it suffices to prove that there exists an increasing sequence (r_m) tending to ∞ such that

$$\lim_{m\to+\infty}\int_s^t\int_{\partial B(0,r_m)} (|\varphi\nabla u| + |u\nabla\varphi|)\, d\tau d\sigma = 0.$$

Indeed, suppose so, and then applying (5.15) for (r_m) and taking limits on m imply (5.12) holds. Let us show the existence of such sequence. Let $0 < \lambda < 1$ be a constant to be determined. Define

$$\begin{aligned}\Phi(R) &:= \int_{\lambda R}^R r^{n-1}\int_s^t\int_{\partial B(0,r)} (|\varphi\nabla u| + |u\nabla\varphi|)\, d\tau d\sigma dr \\ &= \int_s^t\int_{\lambda R<|x|<R} (|\varphi\nabla u| + |u\nabla\varphi|)\, d\tau dx,\end{aligned}$$

[1]By this we mean the energy estimates obtained by multiplying u with proper cut-off and seeing this as a solution to the heat equation with localized source term.

and denote by $\Phi_i(R)$ the i-th term for $i = 1, 2$. Then, it is enough to show that $\Phi(R)$ is bounded. For $\Phi_1(R)$, Caccioppoli's inequality, kernel representation of the heat semigroup, and (i) altogether imply

$$\begin{aligned}
\Phi_1(R) &\leq \left(\int_s^t \int_{\lambda R<|x|<R} |\nabla u|^2 \right)^{1/2} \left(\int_s^t \int_{\lambda R<|x|<R} |\varphi|^2 \right)^{1/2} \\
&\lesssim_\kappa \left[\frac{1}{s-a} \left(1 + \frac{t-a}{R^2} \right) \int_a^b \int_{|x|<\kappa R} |u|^2 \right]^{1/2} \left(\int_s^t e^{-\frac{2cd^2}{t-\tau}} \, d\tau \right)^{1/2} \|h\|_2 \\
&\lesssim_{a,b,\kappa} \left(\frac{t-s}{s-a} \right)^{1/2} \exp\left(-\frac{cd^2}{t-s} + \frac{\gamma(\kappa R)^2}{b-a} \right) \|h\|_2
\end{aligned}$$

for sufficiently large R. Here, $\kappa > 1$ and $0 < c < 1/4$ are constants to be determined, and $d := \operatorname{dist}(\operatorname{supp}(h), \{\lambda R < |x| < R\})$. Then, pick κ close to 1, λ close to 1, and c close to $1/4$ so that $\gamma < c\lambda^2/\kappa^2$, and thus

$$\gamma < \frac{cd^2}{(\kappa R)^2} < \frac{cd^2}{(\kappa R)^2} \cdot \frac{b-a}{t-s} \tag{5.17}$$

for sufficiently large R. We hence conclude that $\Phi_1(R)$ is bounded.

For $\Phi_2(R)$, the kernel representation of $(t^{1/2}\nabla e^{t\Delta})_{t>0}$ and (i) yield

$$\begin{aligned}
\Phi_2(R) &\leq \left(\int_a^b \int_{\lambda R<|x|<R} |u|^2 \right)^{1/2} \left(\int_s^t \int_{\lambda R<|x|<R} |\nabla \varphi|^2 \right)^{1/2} \\
&\lesssim e^{\frac{\gamma R^2}{b-a}} \left(\int_s^t e^{-\frac{2cd^2}{t-\tau}} \frac{d\tau}{t-\tau} \right)^{1/2} \|h\|_2 \\
&\lesssim \frac{(t-s)^{1/2}}{c^{1/2} d} \exp\left(-\frac{cd^2}{t-s} + \frac{\gamma R^2}{b-a} \right) \|h\|_2.
\end{aligned}$$

The same choice for λ and c as above implies (5.17), and hence the boundedness of $\Phi_2(R)$. This completes the proof. □

Let us recall some topological facts about tempered distributions.

Lemma 5.13. *Let X be a Fréchet space and Y be a normed space. Let I be an index set and $\{\psi_\alpha\}_{\alpha \in I}$ be a collection of continuous linear maps from X to Y. If $\sup_{\alpha \in I} \|\psi_\alpha(x)\|_Y$ is bounded for any $x \in X$, then the family $\{\psi_\alpha\}_{\alpha \in I}$ is equicontinuous.*

Proof. It is a direct consequence of a generalized version of Banach-Steinhaus theorem on barrelled spaces, due to the fact that Fréchet spaces are barrelled spaces, see [Bou87, §III.4]. □

One can easily obtain two corollaries. The pairings below are all understood in the sense of tempered distributions.

Corollary 5.14. *Let (φ_k) be a sequence converging to φ in $\mathscr{S}(\mathbb{R}^n)$ and (u_k) be a sequence converging to u in $\mathscr{S}'(\mathbb{R}^n)$. Then $(\langle u_k, \varphi_k\rangle)$ converges to $\langle u, \varphi\rangle$.*

Corollary 5.15. *Any bounded sequence in $\mathscr{S}'(\mathbb{R}^n)$ has a convergent subsequence.*

Let us provide the proof of Theorem 5.7.

Proof of Theorem 5.7. Given $0 < s < t < T$, pick a, b so that $0 < a < s < t < b < T$. Applying Lemma 5.12, we get for any $h \in C_c^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} u(t,x)h(x)dx = \int_{\mathbb{R}^n} u(s,x)(e^{(t-s)\Delta}h)(x)dx. \tag{5.18}$$

Moreover, Corollary 5.15 implies there is a subsequence (t_{k_j}) so that $(u(t_{k_j}))$ converges to some u_0 in $\mathscr{S}'(\mathbb{R}^n)$ as $j \to \infty$. One can also easily verify that $e^{(t-s)\Delta}h$ converges to $e^{t\Delta}h$ in $\mathscr{S}(\mathbb{R}^n)$ as $s \to 0$. Thus, we infer from Corollary 5.14 that

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} u(t_{k_j}, x) \left(e^{(t-t_{k_j})\Delta}h\right)(x)dx = \left\langle u_0, e^{t\Delta}h\right\rangle.$$

Applying (5.18) for $s = t_{k_j}$ and taking limits on j yield $u(t) = e^{t\Delta}u_0$ in $\mathscr{D}'(\mathbb{R}^n)$. As the right-hand side belongs to $\mathscr{S}'(\mathbb{R}^n)$, so does $u(t)$ for all $0 < t < T$. In particular, it implies $u(t)$ converges to u_0 in $\mathscr{S}'(\mathbb{R}^n)$ as $t \to 0$, so u_0 is unique. This completes the proof. $\square$

5.3.3 Applications

Our statement covers many functional spaces of common use in analysis, even when one expects that the initial data is a distribution. Let us illustrate the result with an example related to the famous work of H. Koch and D. Tataru on Navier-Stokes equations [KT01]. Let BMO^{-1} be the collection of distributions $f \in \mathscr{D}'(\mathbb{R}^n)$ with $f = \operatorname{div} g$ for some $g \in \mathrm{BMO}(\mathbb{R}^n; \mathbb{C}^n)$. The space BMO^{-1} can be embedded into $\mathscr{S}'(\mathbb{R}^n)$. It is isomorphic to the homogeneous Triebel–Lizorkin space $\dot{F}^{-1}_{\infty,2}$, with equivalent norms, see, *e.g.*, [Tri83, §5.1]. Moreover, a well-known characterization of BMO^{-1} is that

$$\|f\|_{\mathrm{BMO}^{-1}} \eqsim \|e^{t\Delta}f\|_{T^\infty}. \tag{5.19}$$

Here T^∞ denotes the tent space T^∞_0. In particular, a tempered distribution f lies in BMO^{-1} if $(e^{t\Delta}f)(x)$ lies in T^∞. Remark that if f belongs to BMO^{-1}, as a function of $t \in [0,\infty)$, $e^{t\Delta}f$ is continuous in BMO^{-1} equipped with its weak star topology (or equipped with the topology inherited from that of $\mathscr{S}'(\mathbb{R}^n)$, by density).

A natural question is whether all T^∞ functions solving the heat equation are of that form. We answer it in the affirmative.

Theorem 5.16. *Given a global distributional solution to the heat equation* $u \in T^\infty$*, there exists a unique* $u_0 \in \mathrm{BMO}^{-1}$ *so that* $u(t) = e^{t\Delta}u_0$ *for any* $t > 0$.

Proof. It suffices to verify the two conditions in Theorem 5.7. Indeed, it shows that there exists a unique $u_0 \in \mathscr{S}'(\mathbb{R}^n)$ so that $u(t) = e^{t\Delta}u_0$ for any $t > 0$. We then get $u_0 \in \mathrm{BMO}^{-1}$ by (5.19) since $(e^{t\Delta}u_0)(x) = u(t,x)$ belongs to T^∞.

Let us verify the conditions. First, (i) readily follows as for $0 < a < b < \infty$,

$$\begin{aligned}\int_a^b \int_{B(0,R)} |u(t,y)|^2 dtdy &= \int_a^b \int_{B(0,R)} |u(t,y)|^2 \left(\frac{1}{|B(y,b^{1/2})|} \int_{B(y,b^{1/2})} dx \right) dtdy \\ &\le \int_{B(0,R+b^{1/2})} \left(\frac{1}{|B(x,b^{1/2})|} \int_a^b \int_{B(x,b^{1/2})} |u(t,y)|^2 dtdy \right) dx \\ &\le \|u\|_{T^\infty}^2 |B(0,R+b^{1/2})|.\end{aligned}$$

Next, we claim that there exists $M > 0$ so that for any $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{R}^n)$,

$$\sup_{0<t<1/2} |\langle u(t), \varphi\rangle| \lesssim \mathcal{P}_M(\varphi)\|u\|_{T^\infty}, \tag{5.20}$$

where $\mathcal{P}_M$ is the norm defined in (5.8) given by

$$\mathcal{P}_M(\varphi) := \sup_{|\alpha|+|\beta|\le M} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \varphi(x)|.$$

For fixed $0 < t < 1/2$, standard considerations allow one to extend $u(t)$ to a tempered distribution so that (5.20) holds for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$, which proves (ii). As for the claim, fix $0 < t < 1/2 < t' < 1$ and let $\varphi \in C_{\mathsf{c}}^\infty(\mathbb{R}^n)$. Using the equation for u and integration by parts,

$$\begin{aligned}|\langle u(t'), \varphi\rangle - \langle u(t), \varphi\rangle| &\le \int_t^{t'} \int_{\mathbb{R}^n} |u(s,x)||\Delta\varphi(x)| dsdx \\ &= \int_t^{t'} \int_{B(0,1)} |u(s,x)||\Delta\varphi(x)| dsdx \\ &\quad + \sum_{k=1}^\infty \int_t^{t'} \int_{2^{k-1}\le|x|<2^k} |u(s,x)||\Delta\varphi(x)| dsdx.\end{aligned}$$

Denote by I_0 the first term and I_k the k-th term in the summation. Cauchy-Schwarz inequality yields

$$\begin{aligned}I_0 &\le |B(0,1)| \left(\sup_{|x|<1} |\Delta\varphi(x)| \right) \left(\frac{1}{|B(0,1)|} \int_0^1 \int_{B(0,1)} |u(s,x)|^2 dsdx \right)^{1/2} \\ &\lesssim_n \mathcal{P}_2(\varphi)\|u\|_{T^\infty}.\end{aligned}$$

Similarly, we also have that

$$
\begin{aligned}
I_k &\lesssim_n 2^{kn} \left(\sup_{2^{k-1} \le |x| < 2^k} |\Delta\varphi(x)| \right) \left(\frac{1}{|B(0,2^k)|} \int_0^1 \int_{B(0,2^k)} |u(s,x)|^2 ds dx \right)^{1/2} \\
&\lesssim_n 2^{-k} \left(\sup_{x \in \mathbb{R}^n} |x|^{n+1} |\Delta\varphi(x)| \right) \|u\|_{T^\infty} \le 2^{-k} \mathcal{P}_{n+3}(\varphi) \|u\|_{T^\infty}.
\end{aligned}
$$

We thus obtain

$$
|\langle u(t'), \varphi \rangle - \langle u(t), \varphi \rangle| \lesssim_n \mathcal{P}_{n+3}(\varphi) \|u\|_{T^\infty}.
$$

Taking average in $t' \in (1/2, 1)$ implies

$$
|\langle u(t), \varphi \rangle| \lesssim_n \int_{1/2}^1 |\langle u(t'), \varphi \rangle| dt' + \mathcal{P}_{n+3}(\varphi) \|u\|_{T^\infty}.
$$

The same argument as above yields

$$
\int_{1/2}^1 |\langle u(t'), \varphi \rangle| dt' \le \int_{1/2}^1 \int_{\mathbb{R}^n} |u(t',x)| |\varphi(x)| dt' dx \lesssim_n \mathcal{P}_{n+1}(\varphi) \|u\|_{T^\infty}.
$$

This completes the proof. □

5.4 Well-posedness and representation

Applying the general representation result (cf. Theorem 5.7) to the framework of tent spaces gives the following main theorem on well-posedness and representation of heat solutions in the class $\nabla u \in T^p_{s/2}$.

Theorem 5.17 (Heat equation and weighted tent spaces)**.** *Let $s \in \mathbb{R}$, $0 < p \le \infty$, and $g \in \mathscr{S}'(\mathbb{R}^n)$.*

(i) (Weighted tent-space estimates) *Suppose $s < 1$. If $g \in \dot{H}^{s,p}$, then the function $(t,x) \mapsto \nabla e^{t\Delta} g(x)$ belongs to $T^p_{s/2}$ with*

$$
\|\nabla e^{t\Delta} g\|_{T^p_{s/2}} \eqsim \|g\|_{\dot{H}^{s,p}}.
$$

(ii) (Representation of heat solutions) *Let u be a distributional solution to the heat equation on $\mathbb{R}^{1+n}_+$ with $\nabla u \in T^p_{s/2}$. Suppose $s > -1$ and $\frac{n}{n+s+1} \le p \le \infty$. Then there exists a unique $u_0 \in \mathscr{S}'$ so that $u(t) = e^{t\Delta} u_0$ for all $t > 0$. Moreover,*

(1) *If $s \ge 1$ and $\frac{n}{n+s-1} \le p \le \infty$, then u is a constant.*

(2) *If $-1 < s < 1$ and $\frac{n}{n+s+1} \le p \le \infty$, then there exist $g \in \dot{H}^{s,p}$ and $c \in \mathbb{C}$ such that $u_0 = g + c$, so $u(t) = e^{t\Delta} g + c$ for all $t > 0$.*

The first point (i) shows a precise correspondence between Sobolev regularity of the initial data and the choice of the power weight. The second point (ii) shows the regularity range $s < 1$ is essentially sharp.

When $s = 0$, the first point is reminiscent to the work of Fefferman and Stein [FS72] on defining Hardy spaces by so-called *conical square functions* when translated to parabolic setting. But their point was to take extensions not related to any kind of equations to obtain an intrinsic definition of Hardy spaces. Here, we have to restrict ourselves to extensions related to the heat equation. It is also reminiscent to Littlewood–Paley theory using rather vertical square functions. The reader can refer to [Tri20, §4.1] for more on this topic.

Remark 5.18. Note that we work in $\mathscr{S}'$, and not just modulo polynomials. This induces restrictions on s and p in the proof.

Combining these two points directly shows the following

Corollary 5.19 (Isomorphism)**.** *Let $-1 < s < 1$ and $\frac{n}{n+s+1} \le p \le \infty$. The map $g \mapsto (e^{t\Delta}g)_{t>0}$ is a bijection from $\dot{H}^{s,p} + \mathbb{C}$ onto the space of distributional solutions u to the heat equation with $\nabla u \in T^p_{s/2}$, and*

$$\|g\|_{\dot{H}^{s,p}} \eqsim \|\nabla u\|_{T^p_{s/2}}.$$

5.4.1 Growth of the local norm

Let us start the proof of Theorem 5.17 by three lemmas on the growth of local L^2-norms of functions controlled by its tent space norm.

Lemma 5.20. *Let $0 < p \le 2$ and $\beta \in \mathbb{R}$. For $0 < a < b < \infty$, T^p_β can be embedded into $L^2_\beta((a,b) \times \mathbb{R}^n) := L^2((a,b), t^{-2\beta}dt; L^2(\mathbb{R}^n))$ with*

$$\|u\|_{L^2_\beta((a,b)\times\mathbb{R}^n)} \lesssim_p a^{-\frac{n}{2p}} b^{\frac{n}{4}} \|u\|_{T^p_\beta}.$$

Proof. It suffices to prove the case $\beta = 0$. For any $R > a^{1/2}$, observe that $(a,b) \times B(0,R) \subset \Gamma_\lambda(x)$ for any $x \in B(0,R)$ with $\lambda := 2Ra^{-1/2} > 1$. Then applying the change of aperture (cf. Lemma 3.12) for $p \le 2$ yields

$$\begin{aligned}
\|u\|_{L^2((a,b)\times B(0,R))} &\le b^{n/4} \left(\int_a^b \int_{B(0,R)} |u(t,y)|^2 \frac{dt}{t^{n/2}} dy \right)^{1/2} \\
&\le b^{n/4} \inf_{x\in B(0,R)} \mathcal{A}^{(\lambda)}_{0;2}(u)(x) \\
&\le b^{n/4} \left(\fint_{B(0,R)} \mathcal{A}^{(\lambda)}_{0;2}(u)(x)^p dx \right)^{1/p} \\
&\lesssim b^{\frac{n}{4}} R^{-\frac{n}{p}} \|u\|_{T^{p;2}_{\lambda,0}} \lesssim b^{\frac{n}{4}} R^{-\frac{n}{p}} \lambda^{\frac{n}{p}} \|u\|_{T^p_0} \lesssim a^{-\frac{n}{2p}} b^{\frac{n}{4}} \|u\|_{T^p_0},
\end{aligned}$$

where $\mathcal{A}_{0;2}^{(\lambda)}$ is defined in (3.1). The controlling constant is independent of R, so we conclude by letting R tend to infinity. $\square$

Lemma 5.21. *Let $0 < p \leq \infty$ and $\beta \in \mathbb{R}$. Let u be in T^p_β. Then for $0 < a < b < \infty$, the function*

$$F(x) := \left(\int_a^b \int_{B(x,b^{1/2})} |u(t,y)|^2 dt dy \right)^{1/2}$$

belongs to $L^p(\mathbb{R}^n)$ with $\|F\|_p \lesssim_{a,b,\beta} \|u\|_{T^p_\beta}$.

Proof. First consider $0 < p < \infty$. Pick $(z_k)_{1 \leq k \leq N}$ in $B(0, b^{1/2})$ such that $B(0,b^{1/2}) \subset \bigcup_{k=1}^N B(z_k, a^{1/2})$. Thus, for any $x \in \mathbb{R}^n$, $B(x,b^{1/2}) \subset \bigcup_{k=1}^N B(x+z_k, a^{1/2})$. We hence get

$$\begin{aligned} F(x) &\leq \sum_{k=1}^N b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \left(\int_a^b \int_{B(x+z_k,a^{1/2})} |t^{-\beta} u(t,y)|^2 \frac{dt}{t^{n/2}} dy \right)^{1/2} \\ &\leq b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \sum_{k=1}^N \left(\int_a^b \int_{B(x+z_k,t^{1/2})} |t^{-\beta} u(t,y)|^2 \frac{dt}{t^{n/2}} dy \right)^{1/2}. \end{aligned}$$

Therefore, by taking L^p-norms, we conclude that

$$\|F\|_p \lesssim_N b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \|u\|_{T^p_\beta}. \tag{5.21}$$

For $p = \infty$, note that $(a,b) \times B(x, b^{1/2}) \subset (0,b) \times B(x,b^{1/2})$, so

$$\begin{aligned} F(x) &\lesssim b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \left(\int_a^b \fint_{B(x,b^{1/2})} |t^{-\beta} u(t,y)|^2 dt dy \right)^{1/2} \\ &\leq b^{\frac{n}{4}} \max\{a^\beta, b^\beta\} \sup_{B : x \in B} \left(\int_0^{r(B)^2} \fint_B |t^{-\beta} u(t,y)|^2 dt dy \right)^{1/2}. \end{aligned}$$

Taking L^∞-norm on both sides implies (5.21). $\square$

Lemma 5.22. *Let $s \in \mathbb{R}$ and $0 < p \leq \infty$. Let u be in $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$ with $\nabla u \in T^p_{s/2}$. Then for $0 < a < b < \infty$ and $R > 1$,*

$$\int_a^b \int_{B(0,R)} |u|^2 \lesssim_{a,b,p,s} R^{3n+2} \left(\|\nabla u\|^2_{T^p_{s/2}} + \|u\|^2_{L^2((a,b) \times B(0,1))} \right). \tag{5.22}$$

Proof. Poincaré's inequality yields

$$\int_{B(0,R)} \left| u - \fint_{B(0,R)} u \right|^2 \lesssim R^2 \int_{B(0,R)} |\nabla u|^2.$$

Here and in the sequel, unspecified measures in the integral are unweighted Lebesgue measure. One can also easily verify that

$$\left| \fint_{B(0,R)} u - \fint_{B(0,1)} u \right| \lesssim R \int_{B(0,R)} |\nabla u|.$$

Then we have

$$\begin{aligned}\int_a^b \int_{B(0,R)} |u|^2 &\lesssim \int_a^b \int_{B(0,R)} \left| u - \fint_{B(0,R)} u \right|^2 \\ &\quad + \int_a^b \int_{B(0,R)} \left| \fint_{B(0,R)} u - \fint_{B(0,1)} u \right|^2 + R^n \int_a^b \fint_{B(0,1)} |u|^2 \\ &\lesssim R^{2n+2} \int_a^b \int_{B(0,R)} |\nabla u|^2 + R^n \int_a^b \int_{B(0,1)} |u|^2.\end{aligned}$$

Then (5.22) follows from the claim that for any $R > 1$,

$$\int_a^b \int_{B(0,R)} |\nabla u|^2 \lesssim_{a,b,p,s} R^n \|\nabla u\|^2_{T^p_{s/2}}. \tag{5.23}$$

Let us verify the claim. For $2 \le p \le \infty$, Lemma 5.21 implies

$$\left\| \left(\int_a^b \fint_{B(\cdot, b^{1/2})} |\nabla u(t,y)|^2 dtdy \right)^{1/2} \right\|_p \lesssim_{a,b,s} \|\nabla u\|_{T^p_{s/2}}.$$

Denote by r the Hölder conjugate of $p/2$, and (5.23) follows from

$$\begin{aligned}\int_a^b \int_{B(0,R)} |\nabla u|^2 &\lesssim \int_{B(0,R+b^{1/2})} dx \int_a^b \fint_{B(x,b^{1/2})} |\nabla u(t,y)|^2 dtdy \\ &\lesssim_{a,b,p,s} R^{\frac{n}{r}} \|\nabla u\|^2_{T^p_{s/2}}.\end{aligned}$$

For $0 < p \le 2$, (5.23) follows by Lemma 5.20 as

$$\int_a^b \int_{B(0,R)} |\nabla u|^2 \lesssim_{a,b,p,s} \|\nabla u\|^2_{T^p_{s/2}}.$$

This completes the proof. □

5.4.2 Proof of Theorem 5.17

We first prepare a technical lemma. Let $N \ge 0$. Recall the norm $\mathcal{P}_N$ the norm on Schwartz functions on $\mathbb{R}^n$ defined in (5.8) by

$$\mathcal{P}_N(\phi) := \sup_{|\alpha|+|\gamma| \le N} \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\gamma \phi(x)|.$$

Lemma 5.23. *Let $\beta \in \mathbb{R}$, $0 < p \leq \infty$, and $\sigma \geq 0$. Let ϕ be a Schwartz function on $\mathbb{R}^n$. Let $0 \leq a < b < \infty$.*

(i) *If $a > 0$, then there exists $N \geq 0$ so that*

$$\|\mathbb{1}_{(a,b)}(t)\phi\|_{T^p_\beta} + \|\mathbb{1}_{(a,b)}(t)\phi\|_{T^\infty_{\beta,(\sigma)}} \lesssim_a (b^{-2\beta+1} - a^{-2\beta+1})^{1/2}\mathcal{P}_N(\phi).$$

(ii) *If $a = 0$, then for $\beta < 1/2$ and $0 \leq \sigma \leq (1-2\beta)/n$, there exists $N \geq 0$ so that*

$$\|\mathbb{1}_{(0,b)}(t)\phi\|_{T^p_\beta} \lesssim b^{-\beta+\frac{1}{2}}\mathcal{P}_N(\phi), \quad \|\mathbb{1}_{(0,b)}(t)\phi\|_{T^\infty_{\beta,(\sigma)}} \lesssim b^{-\beta+\frac{1}{2}-\frac{n}{2}\sigma}\mathcal{P}_0(\phi).$$

Proof. The proof follows by straightforward computation. For sake of completeness, we provide the details. Denote by $\langle x \rangle$ the Japanese bracket given by $\langle x \rangle := (1+|x|^2)^{1/2}$.

First consider (i). For $0 < p < \infty$, pick $N > n/p$ and we have

$$\begin{aligned}\|\mathbb{1}_{(a,b)}(t)\phi\|_{T^p_\beta} &\lesssim \left(\int_a^b t^{-2\beta}dt\right)^{1/2} \left(\int_{\mathbb{R}^n} \langle x \rangle^{-Np} \left(\sup_{y \in B(x,t^{1/2})} \langle y \rangle^N |\phi(y)|\right)^p dx\right)^{1/p} \\ &\lesssim (b^{-2\beta+1} - a^{-2\beta+1})^{1/2}\mathcal{P}_N(\phi)\end{aligned}$$

as desired, where the converges is ensured as $N > n/p$.

For $p = \infty$ and $\sigma \geq 0$, pick $N \geq 0$. For $B \subset \mathbb{R}^n$ as a ball, once $(0, r(B)^2) \times B$ intersects with $(a,b) \times \mathbb{R}^n$, we get $r(B)^2 \geq a$, so

$$\begin{aligned}\frac{1}{|B|^{2\sigma}} \int_0^{r(B)^2} \fint_B |t^{-\beta}\mathbb{1}_{(a,b)}(t)\phi(y)|^2 dtdy &\lesssim a^{-n\sigma}\mathcal{P}_0(\phi)^2 \int_a^b t^{-2\beta}dt \\ &\lesssim_a (b^{-2\beta+1} - a^{-2\beta+1})\mathcal{P}_0(\phi)^2.\end{aligned}$$

Taking supremum over all the balls B gives the estimates desired.

Next, consider (ii). The first inequality follows from the same arguments as in (i), where the convergence is ensured as $\beta < 1/2$, *i.e.*, $-2\beta > -1$. To prove the second inequality, we fix a ball $B \subset \mathbb{R}^n$. Note that the condition that $(0, r(B)^2) \times B$ intersects with $(0,b) \times \mathbb{R}^n$ requires $r(B)^2 \leq b$. For such ball B, we have

$$\begin{aligned}\frac{1}{|B|^{2\sigma}} \int_0^{r(B)^2} \fint_B |t^{-\beta}\mathbb{1}_{(0,b)}(t)\phi(y)|^2 dtdy &\lesssim \mathcal{P}_0(\phi)^2 r(B)^{-2\beta+1-2n\sigma} \\ &\lesssim b^{-2\beta+1-n\sigma}\mathcal{P}_0(\phi)^2.\end{aligned}$$

In the last inequality, we use $-2\beta+1-n\sigma \geq 0$. This completes the proof. $\square$

To finish this chapter, let us present the proof of Theorem 5.17.

Proof of Theorem 5.17. The statement (i) has been shown in Corollary 5.6. Let us focus on (ii). Let $s > -1$ and $\frac{n}{n+s+1} \leq p \leq \infty$. Let u be a distributional (hence smooth) solution to the heat equation on $\mathbb{R}^{1+n}_+$ with $\nabla u \in T^p_{s/2}$.

Let us first prove that there exists $u_0 \in \mathscr{S}'$ so that $u(t) = e^{t\Delta} u_0$ for all $t > 0$. Our strategy is to apply Theorem 5.7. Let us verify the size condition and the uniform control condition there. Recall that u is smooth. Lemma 5.22 implies that for $0 < a < b < \infty$, there exists $\gamma \in (0, 1/4)$ so that for any $R > 0$,

$$\left(\int_a^b \int_{B(0,R)} |u|^2 \right)^{1/2} \lesssim_{a,b,\gamma} \exp\left(\frac{\gamma R^2}{b-a} \right),$$

which is exactly the size condition. For the uniform control condition, we show that there exists $N > 0$ so that for any $\phi \in C^\infty_c(\mathbb{R}^n)$,

$$\sup_{0<t<1/2} |\langle u(t), \phi \rangle| \lesssim \mathcal{P}_N(\phi) \left(\|\nabla u\|_{T^p_{s/2}} + \|u\|_{L^2((1,2)\times B(0,1))} \right). \tag{5.24}$$

To this end, fix $0 < t < 1/2 < 1 < t' < 2$ and $\phi \in C^\infty_c(\mathbb{R}^n)$. Note that

$$\int_1^2 \int_{\mathbb{R}^n} |u||\phi| \leq \int_1^2 \int_{|x|<1} |u||\phi| + \sum_{k=1}^\infty \int_1^2 \int_{2^{k-1} \leq |x| < 2^k} |u||\phi|.$$

Denote by I_k the k-th term for $k \geq 0$. For $k = 0$, we have

$$I_0 \lesssim \|u\|_{L^2((1,2)\times B(0,1))} \mathcal{P}_0(\phi).$$

For $k \geq 1$, using (5.22), we get

$$\begin{aligned} I_k &\leq \left(\int_1^2 \int_{|x|<2^k} |u|^2 \right)^{1/2} \left(\int_1^2 \int_{2^{k-1} \leq |x| < 2^k} |\phi|^2 \right)^{1/2} \\ &\lesssim 2^{k(2n+1)} \left(\|\nabla u\|_{T^p_{s/2}} + \|u\|_{L^2((1,2)\times B(0,1))} \right) \sup_{2^{k-1} \leq |x| < 2^k} |\phi(x)| \\ &\lesssim 2^{-k} \mathcal{P}_{N_1}(\phi) \left(\|\nabla u\|_{T^p_{s/2}} + \|u\|_{L^2((1,2)\times B(0,1))} \right) \end{aligned}$$

for some $N_1 > 0$. We hence obtain

$$\int_1^2 \int_{\mathbb{R}^n} |u||\phi| \lesssim \mathcal{P}_{N_1}(\phi) \left(\|\nabla u\|_{T^p_{s/2}} + \|u\|_{L^2((1,2)\times B(0,1))} \right).$$

Next, we claim that there exists $N_2 > 0$ so that

$$\int_0^2 \int_{\mathbb{R}^n} |\nabla u||\nabla \phi| \lesssim \mathcal{P}_{N_2}(\phi) \|\nabla u\|_{T^p_{s/2}}. \tag{5.25}$$

Then (5.24) follows by using the mean-value inequality in time and the equation for u. Indeed, pick $N > \max\{N_1, N_2\}$, and we get

$$\begin{aligned}
|\langle u(t), \phi\rangle| &\le \fint_1^2 |\langle u(t'), \phi\rangle| \, dt' + \fint_1^2 |\langle u(t'), \phi\rangle - \langle u(t), \phi\rangle| \, dt' \\
&\le \fint_1^2 |\langle u(t'), \phi\rangle| \, dt' + \int_0^2 \int_{\mathbb{R}^n} |\nabla u||\nabla \phi| \\
&\lesssim \mathcal{P}_N(\phi) \left(\|\nabla u\|_{T^p_{s/2}} + \|u\|_{L^2((1,2)\times B(0,1))} \right)
\end{aligned}$$

as desired.

Let us verify (5.25). We use duality of tent spaces. Let $s > -1$. For $1 < p \le \infty$, as $-s/2 < 1/2$, Lemma 5.23 (ii) implies that there exists $N_2 > 0$ so that

$$\|\mathbb{1}_{(0,2)} \nabla \phi\|_{T^{p'}_{-s/2}} \lesssim \mathcal{P}_{N_2}(\phi).$$

Since $T^p_{s/2}$ identifies with the dual of $T^{p'}_{-s/2}$ via $L^2(\mathbb{R}^{1+n}_+)$-duality (see Proposition 3.3), we get

$$\int_0^2 \int_{\mathbb{R}^n} |\nabla u||\nabla \phi| \lesssim \|\nabla u\|_{T^p_{s/2}} \|\mathbb{1}_{(0,2)} \nabla \phi\|_{T^{p'}_{-s/2}} \lesssim \|\nabla u\|_{T^p_{s/2}} \mathcal{P}_{N_2}(\phi)$$

as desired. For $\frac{n}{n+s+1} \le p \le 1$, *i.e.*, $[p,1] = \frac{1}{p} - 1 \le (1+s)/n$, Lemma 5.23 (ii) also yields

$$\|\mathbb{1}_{(0,2)} \nabla \phi\|_{T^\infty_{-s/2,([p,1])}} \lesssim \mathcal{P}_0(\phi) \le \mathcal{P}_{N_2}(\phi).$$

The $L^2(\mathbb{R}^{1+n}_+)$-duality identifies $T^\infty_{-s/2,([p,1])}$ with the dual of $T^p_{s/2}$, so we obtain

$$\int_0^2 \int_{\mathbb{R}^n} |\nabla u||\nabla \phi| \lesssim \|\nabla u\|_{T^p_{s/2}} \|\mathbb{1}_{(0,2)} \nabla \phi\|_{T^\infty_{-s/2,([p,1])}} \lesssim \|\nabla u\|_{T^p_{s/2}} \mathcal{P}_{N_2}(\phi)$$

as desired. This proves (5.25).

Therefore, we obtain a (hence unique) $u_0 \in \mathscr{S}'$ so that $u(t) = e^{t\Delta} u_0$ for all $t > 0$. Then Corollary 5.6 and Proposition 5.3 show the desired properties (1) and (2) for u_0. This completes the proof. $\square$

5.5 Results for homogeneous Besov spaces

One can also study the case where the initial data are taken in homogeneous Besov spaces $\dot{B}^s_{p,p}$. The definition of $\dot{B}^s_{p,p}$ can be analogously adapted from Definition 5.2, as "realization" of homogeneous Besov spaces defined on $\mathscr{S}'/\mathscr{P}$. In fact, one can also take the definition of $\dot{B}^s_{p,p}$ as in [BCD11, Definition 2.15], which does not contain polynomials for any s and p. See [BCD11, Remark 2.26] for more detailed discussion.

The counterparts of tent spaces are Z-spaces (see Definition 3.7). Recall that in the parabolic settings (i.e., homogeneity $m=2$), for any $p \in (0,\infty)$ and $\beta \in \mathbb{R}$, the *(parabolic) Z-space* Z^p_β consists of measurable functions F on $\mathbb{R}^{1+n}_+$ for which the (quasi-)norm

$$\|F\|_{Z^p_\beta} := \left(\int_{\mathbb{R}^{1+n}_+} \left(\int_{t/2}^t \fint_{B(x,t^{1/2})} |s^{-\beta} F(s,y)|^2 ds dy \right)^{p/2} \frac{dt}{t} dx \right)^{1/p} < \infty.$$

For $p=\infty$, we set

$$\|F\|_{Z^\infty_\beta} := \sup_{t>0, x\in\mathbb{R}^n} \left(\int_{t/2}^t \fint_{B(x,t^{1/2})} |s^{-\beta} F(s,y)|^2 ds dy \right)^{1/2}.$$

The relation between $\dot{B}^s_{p,p}$ (resp. Z^p_β) and $\dot{H}^{s,p}$ (resp. T^p_β) is given by real interpolation. Let $0 < p_0, p_1 \le \infty$ and $\theta \in (0,1)$. Let $s_0, s_1, \beta_0, \beta_1 \in \mathbb{R}$ with $s_0 \neq s_1$ and $\beta_0 \neq \beta_1$. Suppose $s, \beta \in \mathbb{R}$ and $p \in (0,\infty]$ with

$$s = (1-\theta)s_0 + \theta s_1, \quad \beta = (1-\theta)\beta_0 + \theta\beta_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then we have

$$(\dot{H}^{s_0,p_0}, \dot{H}^{s_1,p_1})_{\theta,p} = \dot{B}^s_{p,p}, \quad (T^{p_0}_{\beta_0}, T^{p_1}_{\beta_1})_{\theta,p} = Z^p_\beta. \tag{5.26}$$

Let us first consider a special case for $s=1$ and $p=\infty$. Recall that we say a distribution g belongs to the homogeneous Sobolev space $\dot{W}^{1,\infty}$ if ∇g is bounded, and $\|g\|_{\dot{W}^{1,\infty}} := \|\nabla g\|_\infty$. Rademacher's theorem asserts $\dot{W}^{1,\infty}$ coincides with the set of Lipschitz continuous functions (up to almost everywhere equality).

Proposition 5.24. (i) *Let $g \in \dot{W}^{1,\infty}$ be a Lipschitz function. Then the function $(t,x) \mapsto \nabla e^{t\Delta} g(x)$ belongs to $Z^\infty_{1/2}$ with*

$$\|\nabla e^{t\Delta} g\|_{Z^\infty_{1/2}} \eqsim \|g\|_{\dot{W}^{1,\infty}}.$$

(ii) *Let u be a distributional solution to the heat equation on $\mathbb{R}^{1+n}_+$ with $\nabla u \in Z^\infty_{1/2}$. Then there exists $u_0 \in \dot{W}^{1,\infty}$ such that $u(t) = e^{t\Delta} u_0$ for all $t>0$.*

Proof. Observe that the space $Z^\infty_{1/2}$ coincides with the parabolic version of the Kenig–Pipher space X^∞ introduced by [KP93], see Remark 3.8. Then we invoke [AMP19, Theorem 5.4] (or see Remark 6.34) to get (i) as

$$\|\nabla e^{t\Delta} g\|_{Z^\infty_{1/2}} = \|e^{t\Delta} \nabla g\|_{X^\infty} \eqsim \|\nabla g\|_{L^\infty} = \|g\|_{\dot{W}^{1,\infty}}.$$

Moreover, it also asserts that any weak solution $G \in X^\infty$ to the heat equation has a trace $g \in L^\infty$ so that $G(t) = e^{t\Delta} g$ for all $t > 0$.

To prove (ii), we claim that such u also has a distributional limit $u_0 \in \mathscr{S}'$ as $t \to 0$, and $u(t) = e^{t\Delta} u_0$ for any $t > 0$. Then applying the above assertion to $\nabla u \in X^\infty$ yields $\nabla u_0 = g$, so u_0 must be (equal almost everywhere to) a Lipschitz function as desired.

The claim follows by a verbatim adaptation of the proof of Theorem 5.17. We just list the main modifications here. In this case, the estimate in Lemma 5.22 that yields the size condition becomes: For $0 < a < b < \infty$ and $R > 1$,

$$\int_a^b \int_{B(0,R)} |u|^2 \lesssim_{a,b} R^{3n+2} \left(\|\nabla u\|^2_{Z^\infty_{1/2}} + \|u\|^2_{L^2((a,b)\times B(0,1))} \right).$$

For the uniform control condition (5.24), it follows from (5.25), which still holds for $N > n+1$ since

$$\int_0^2 \int_{\mathbb{R}^n} |\nabla u||\nabla \phi| \lesssim \|\nabla u\|_{Z^\infty_{1/2}} \|\mathbb{1}_{(0,2)}\phi\|_{Z^1_{-1/2}} \lesssim \|\nabla u\|_{Z^\infty_{1/2}} \mathcal{P}_N(\phi).$$

Therefore, applying Theorem 5.7 again provides the trace $u_0 \in \mathscr{S}'$ as wanted. This completes the proof. □

We summarize the results in the following

Theorem 5.25 (Heat equation and Z-spaces). *Let $s \in \mathbb{R}$, $0 < p \le \infty$, and $g \in \mathscr{S}'(\mathbb{R}^n)$.*

(i) (Weighted Z-space estimates) *Suppose $s < 1$. If $g \in \dot{B}^s_{p,p}$, then the function $(t,x) \mapsto \nabla e^{t\Delta} g(x)$ belongs to $Z^p_{s/2}$ with*

$$\|\nabla e^{t\Delta} g\|_{Z^p_{s/2}} \eqsim \|g\|_{\dot{B}^s_{p,p}}.$$

The equivalence also holds for $s = 1$, $p = \infty$, and $g \in \dot{W}^{1,\infty}$.

(ii) (Representation of heat solutions) *Let u be a distributional solution to the heat equation on $\mathbb{R}^{1+n}_+$ with $\nabla u \in Z^p_{s/2}$. Suppose $s > -1$ and $\frac{n}{n+s+1} \le p \le \infty$. Then there exists a unique $u_0 \in \mathscr{S}'$ so that $u(t) = e^{t\Delta} u_0$ for all $t > 0$. Moreover,*

(1) *If $s \ge 1$, $\frac{n}{n+s-1} \le p \le \infty$, but $(s,p) \ne (1,\infty)$, then u is a constant.*

(2) *If $s = 1$ and $p = \infty$, then $u_0 \in \dot{W}^{1,\infty}$.*

(3) *If $-1 < s < 1$ and $\frac{n}{n+s+1} \le p \le \infty$, then there exist $g \in \dot{B}^s_{p,p}$ and $c \in \mathbb{C}$ such that $u_0 = g + c$, so $u(t) = e^{t\Delta} g + c$ for all $t > 0$.*

Proof. We just provide the key ingredients of the proof. Detailed verification is left to the reader.

First consider (i). For $s < 1$, one uses the real interpolation (5.26). For $s = 1$ and $p = \infty$, it has been shown in Proposition 5.24 (i).

Next, consider (ii). The existence of the trace follows from a verbatim adaptation of the proof of Theorem 5.17, as is shown in Proposition 5.24 (ii). Then we show the properties of the trace.

For (1), when $s \geq 1$, $\frac{n}{n+s+1} \leq p \leq \infty$ but $(s,p) \neq (1,\infty)$, one can imitate the proof of Corollary 5.6 (ii), using interpolation and direct computation, to get that for $0 < a < 1$ and $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\fint_a^{2a} |\langle \nabla e^{t\Delta} u_0, \phi \rangle| \, dt \lesssim_\phi \| \mathbb{1}_{(a,2a)} \nabla \mathcal{E}_{-\Delta}(u_0) \|_{Z^p_{s/2}},$$

which tends to 0 as $a \to 0$, so $\nabla u_0 = 0$ and u_0 is a constant.

For (2), this is Proposition 5.24 (ii).

For (3), when $-1 < s < 1$ and $\frac{n}{n+s+1} \leq p \leq \infty$, we apply interpolation for the map $\nabla u \mapsto \nabla u_0$ to obtain $\nabla u_0 \in \dot{B}^{s-1}_{p,p}$ with $\|\nabla u_0\|_{\dot{B}^{s-1}_{p,p}} \lesssim \|\nabla u\|_{Z^p_{s/2}}$. Therefore, by lifting property, there exist some $g \in \dot{B}^s_{p,p}$ and $c \in \mathbb{C}$ so that $u_0 = g + c$. This completes the proof. $\square$

Chapter 6

Autonomous parabolic Cauchy problems

"饱食而遨游，泛若不系之舟"[1]

《庄子·列御寇》庄子

The main objective of this chapter is to present a complete picture for existence, uniqueness, and representation of weak solutions (see Definition 6.2) to the Cauchy problem

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(x)\nabla_x u) = f + \operatorname{div}_x F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases}, \tag{6.1}$$

using the framework of tent spaces. We consider time-independent, uniformly elliptic, bounded measurable coefficients.

Assumption 6.1. Throughout this chapter, we assume that the coefficient matrix $A = A(x) \in L^\infty(\mathbb{R}^n; \mathrm{Mat}_n(\mathbb{C}))$ is *uniformly elliptic*, that is, there are $\Lambda_0, \Lambda_1 > 0$ so that for a.e. $x \in \mathbb{R}^n$ and any $\xi, \eta \in \mathbb{C}^n$,

$$\Re(\langle A(x)\xi, \xi\rangle) \geq \Lambda_0 |\xi|^2, \quad |\langle A(x)\xi, \eta\rangle| \leq \Lambda_1 |\xi||\eta|. \tag{6.2}$$

In contrast with the classical maximal regularity theory, we treat initial data which are not necessarily trace spaces (in the sense of real interpolation theory) and carry regularity information by control of the gradient of weak solutions in a *weighted* tent space, where the weight is a power of the time variable. Indeed, we shall show that our approach gives access to initial data

[1]English translation: *Having eaten his fill, he wanders aimlessly, like a boat unmoored.* From Lie Yukou, *Zhuangzi*, Zhuang Zhou

in homogeneous Hardy–Sobolev spaces $\dot{H}^{s,p}$ defined by Littlewood–Paley decomposition (see Section 5.1). The regularity range is between -1 and 1. Our method also applies for initial data in homogeneous Besov spaces.

This chapter is a compilation of (part of) the two articles:

- "*On well-posedness and maximal regularity for parabolic Cauchy problems on weighted tent spaces*" [AH25a], written in collaboration with Pascal Auscher and published in *Journal of Evolution Equations*.
- "*On well-posedness for parabolic Cauchy problems of Lions type with rough initial data*" [AH25b], written in collaboration with Pascal Auscher and published in *Mathematische Annalen*.

6.1 Basic facts about weak solutions

In this section, we review the definition of weak solutions, prove *a priori* energy inequalities, and recall elements of the L^2-theory. Denote by $W^{1,2}$ the inhomogeneous Sobolev space with the norm $\|f\|_{W^{1,2}} := \|f\|_2 + \|\nabla f\|_2$.

Definition 6.2 (Weak solutions)**.** Let $0 \le a < b \le \infty$, Ω be an open subset of $\mathbb{R}^n$, and $Q := (a,b) \times \Omega$. Let f and F be in $\mathscr{D}'(Q)$. A function $u \in L^2_{\mathsf{loc}}((a,b); W^{1,2}_{\mathsf{loc}}(\Omega))$ is called a *weak solution* to the equation

$$\partial_t u - \operatorname{div}(A\nabla u) = f + \operatorname{div} F$$

with *source term* $f + \operatorname{div} F$, if for any $\phi \in C^\infty_{\mathsf{c}}(Q)$,

$$-\iint_Q u\ \partial_t \phi + \iint_Q (A\nabla u) \cdot \nabla \phi = (f, \phi) - (F, \nabla \phi). \tag{6.3}$$

The pairs on the right-hand side are understood as pairings of distributions and test functions on Q. We say u is a *global weak solution* if (6.3) holds for $Q = \mathbb{R}^{1+n}_+$.

Let $a = 0$ and $u_0 \in \mathscr{D}'(\Omega)$. By the *initial condition* $u(0) = u_0$, we further impose that the weak solution $u(t)$ converges to u_0 in $\mathscr{D}'(\Omega)$ as $t \to 0$.

Similarly, there is a corresponding definition of weak solutions for the dual backward equation $-\partial_t u - \operatorname{div}(A^*\nabla u) = f + \operatorname{div} F$.

6.1.1 Energy inequalities

Let us recall without proof a form of the classical Caccioppoli's inequality.

Lemma 6.3 (Caccioppoli's inequality)**.** *Let $0 < a < b < \infty$, $B \subset \mathbb{R}^n$ be a ball, and f, F be in $L^2((a,b) \times 2B)$. Let $u \in L^2((a,b); W^{1,2}(2B))$ be a weak* *solution to the equation* $\partial_t u - \operatorname{div}(A\nabla u) = f + \operatorname{div} F$ *in* $(a,b) \times 2B$. *Then* u *lies in* $C([a,b];L^2(B))$ *with*

$$\|u(b)\|_{L^2(B)}^2 \lesssim \left(\frac{1}{r(B)^2} + \frac{1}{b-a}\right) \int_a^b \|u(s)\|_{L^2(2B)}^2\,ds + r(B)^2 \int_a^b \|f(s)\|_{L^2(2B)}^2\,ds + \int_a^b \|F(s)\|_{L^2(2B)}^2\,ds.$$

Moreover, for any $c \in (a,b)$, *it holds that*

$$\int_c^b \|\nabla u(s)\|_{L^2(B)}^2\,ds \lesssim \frac{1}{c-a}\left(1 + \frac{b-a}{r(B)^2}\right) \int_a^b \|u(s)\|_{L^2(2B)}^2\,ds + \frac{r(B)^2(b-a)}{c-a} \int_a^b \|f(s)\|_{L^2(2B)}^2\,ds + \frac{b-a}{c-a} \int_a^b \|F(s)\|_{L^2(2B)}^2\,ds.$$

The implicit constants are independent of a, b, c, *and* B.

There is also a corresponding version for weak solutions to the backward equation. In the sequel, we refer to “Caccioppoli’s inequality” in both cases.

For any $(t,x) \in \mathbb{R}^{1+n}_+$, the set $W(t,x) := (t,2t) \times B(x,t^{1/2})$ is called a *(parabolic) Whitney cube* at (t,x).

Corollary 6.4. *Let* $\beta \in \mathbb{R}$ *and* $0 < p \leq \infty$. *For any global weak solution* u *on* $\mathbb{R}^{1+n}_+$ *to* $\partial_t u - \operatorname{div}(A\nabla u) = f + \operatorname{div} F$, *the following a priori energy inequality holds*

$$\|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|u\|_{T^p_{\beta+1}} + \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^p_\beta}. \tag{6.4}$$

The same inequality occurs for global weak solutions to the backward equation.

Proof. When $p < \infty$, using Caccioppoli’s inequality on local Whitney cubes and the change of aperture property of tent-space norms, we get

$$\begin{aligned}
\|\nabla u\|_{T^p_{\beta+1/2}} &\eqsim \left(\int_{\mathbb{R}^n}\left(\int_0^\infty t^{-2\beta-1}dt \fint_{t/2}^t \fint_{B(x,t^{1/2})} |\nabla u|^2\right)^{p/2} dx\right)^{1/p} \\
&\lesssim \left(\int_{\mathbb{R}^n}\left(\int_0^\infty t^{-2\beta-2}dt \fint_{t/4}^t \fint_{B(x,2t^{1/2})} |u|^2\right)^{p/2} dx\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}^n}\left(\int_0^\infty t^{-2\beta-1}dt \fint_{t/4}^t \fint_{B(x,2t^{1/2})} |F|^2\right)^{p/2} dx\right)^{1/p} \\
&\quad + \left(\int_{\mathbb{R}^n}\left(\int_0^\infty t^{-2\beta}dt \fint_{t/4}^t \fint_{B(x,2t^{1/2})} |f|^2\right)^{p/2} dx\right)^{1/p} \\
&\lesssim \|u\|_{T^p_{\beta+1}} + \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^p_\beta}.
\end{aligned}$$

When $p = \infty$, we take a covering of the Carleson boxes $(0, r(B)^2) \times B$ by local Whitney cubes and apply Caccioppoli's inequality on each, noting that the enlarged local Whitney cubes are contained in $(0, 16r(B)^2) \times 4B$ with bounded overlapping. Detailed verification is left to the reader. □

6.1.2 L^2-theory

In this section, we summarize the L^2-theory as a starting point. The weak solutions are given by Duhamel's formula and semigroup theory. Define the operator

$$\boxed{L := -\operatorname{div}(A(x)\nabla)}$$

on $L^2(\mathbb{R}^n)$ with domain

$$D(L) := \{f \in W^{1,2}(\mathbb{R}^n) : \operatorname{div}(A\nabla f) \in L^2(\mathbb{R}^n)\}.$$

Note that $-L$ is a maximal accretive operator on L^2, so particularly $-L$ is a densely-defined closed operator on L^2 that generates a bounded analytic semigroup $(e^{-tL})_{t\geq 0}$ on L^2. Define the *semigroup solution map* $\mathcal{E}_L$ from $L^2(\mathbb{R}^n)$ to $L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$ by

$$\mathcal{E}_L(u_0)(t,x) := (e^{-tL}u_0)(x), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n. \tag{6.5}$$

To treat source terms, we also define the *Duhamel operator* $\mathcal{L}_1^L$ from $L^2(\mathbb{R}^{1+n}_+)$ to $L^2_{\mathsf{loc}}((0,\infty); L^2)$ by the L^2-valued Bochner integrals

$$\mathcal{L}_1^L(f)(t) := \int_0^t e^{-(t-s)L} f(s)\, ds, \quad t > 0. \tag{6.6}$$

and the *Lions operator* $\mathcal{R}^L_{1/2}$ from $L^2(\mathbb{R}^{1+n}_+)$ to $L^2_{\mathsf{loc}}((0,\infty); L^2)$ by the L^2-valued Bochner integrals

$$\mathcal{R}^L_{1/2}(F)(t) := \int_0^t e^{-(t-s)L} \operatorname{div} F(s)\, ds, \quad t > 0. \tag{6.7}$$

Proposition 6.5 (L^2-theory)**.** *Let $u_0 \in L^2$ and $F \in L^2(\mathbb{R}^{1+n}_+)$. Then there exists a unique global weak solution u to the Cauchy problem*

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F \\ u(0) = u_0 \end{cases}$$

with $\nabla u \in L^2(\mathbb{R}^{1+n}_+)$. Moreover,

(i) *u belongs to $C_0([0,\infty); L^2)$ with the estimate*

$$\sup_{t\geq 0} \|u(t)\|_2 + \|\nabla u\|_{L^2(\mathbb{R}^{1+n}_+)} \lesssim \|u_0\|_2 + \|F\|_{L^2(\mathbb{R}^{1+n}_+)}.$$

(ii) (Duhamel's formula) $u = \mathcal{E}_L(u_0) + \mathcal{R}^L_{1/2}(F)$.

Remark 6.6. Notice that it does not include the source term f. In fact, as we shall see in Section 6.2, for $f \in L^2(\mathbb{R}^{1+n}_+)$, Duhamel's solution $\mathcal{L}^L_1(f)$ does not fit in this energy class $L^\infty((0,\infty); L^2) \cap L^2((0,\infty); \dot{W}^{1,2})$. It will be separately dealt in Section 6.2.2.

Existence originates from the work of J.-L. Lions [Lio57, Théorème II.3.1], and uniqueness in this larger (likely the largest) class is established in [AMP19, Theorem 3.11]. The reader can refer to [AP25, Theorem 2.2] for a detailed survey of different proofs of this theorem.

This also allows us to obtain several equivalent expressions of these operators, using different interpretations of the equation. This will come into play when studying their extensions. Denote by $\mathbb{I}$ the identity matrix in $\mathrm{Mat}_n(\mathbb{C})$.

Corollary 6.7 (Explicit formulae)**.** *Let $u_0 \in L^2$ and $F \in L^2(\mathbb{R}^{1+n}_+)$.*

(i) *Define $\widetilde{F} := (A - \mathbb{I})\nabla\mathcal{R}^L_{1/2}(F) + F$ on $\mathbb{R}^{1+n}_+$. Then*

$$\mathcal{R}^L_{1/2}(F) = \mathcal{R}^{-\Delta}_{1/2}(\widetilde{F}) = \operatorname{div}\mathcal{L}^{-\Delta}_1(\widetilde{F}) \quad \text{in } \mathscr{D}'(\mathbb{R}^{1+n}_+). \tag{6.8}$$

(ii) *It holds that*

$$\begin{aligned} \mathcal{E}_L(u_0) &= \mathcal{E}_{-\Delta}(u_0) + \mathcal{R}^L_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_{-\Delta}(u_0)) && (6.9)\\ &= \mathcal{E}_{-\Delta}(u_0) + \mathcal{R}^{-\Delta}_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_L(u_0)). && (6.10)\end{aligned}$$

Proof. First consider (i). Proposition 6.5 says that $u := \mathcal{R}^L_{1/2}(F)$ is a global weak solution to the Cauchy problem

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F \\ u(0) = 0 \end{cases}, \tag{L}$$

with $\nabla u \in L^2(\mathbb{R}^{1+n}_+)$. We hence infer that $\widetilde{F}$ lies in $L^2(\mathbb{R}^{1+n}_+)$, and then $\widetilde{u} := \mathcal{R}^{-\Delta}_{1/2}(\widetilde{F})$ is a global weak solution to the Cauchy problem

$$\begin{cases} \partial_t\widetilde{u} - \Delta\widetilde{u} = \operatorname{div}\widetilde{F} = \operatorname{div}((A-\mathbb{I})\nabla u) + \operatorname{div} F \\ \widetilde{u}(0) = 0 \end{cases}$$

with $\nabla\widetilde{u} \in L^2(\mathbb{R}^{1+n}_+)$. Therefore, $w := u - \widetilde{u}$ is a global weak solution to the heat equation with null source term and null initial data. Since $\nabla w \in L^2(\mathbb{R}^{1+n}_+)$, we deduce from uniqueness in Proposition 6.5 for the heat equation that $w = 0$, proving the first equality in (6.8). For the last, as $\widetilde{F} \in L^2(\mathbb{R}^{1+n}_+)$, for a.e. $t > 0$, we have

$$\begin{aligned}\mathcal{R}^{-\Delta}_{1/2}(\widetilde{F})(t) &= \int_0^t e^{(t-s)\Delta} \operatorname{div} \widetilde{F}(s)\, ds \\ &= \operatorname{div} \int_0^t e^{(t-s)\Delta} \widetilde{F}(s)\, ds = \operatorname{div} \mathcal{L}^{-\Delta}_1(\widetilde{F})(t)\end{aligned}$$

in the sense of distributions in $\mathbb{R}^n$ and (i) follows.

Next, we consider (ii). Proposition 6.5 asserts that $v := \mathcal{E}_L(u_0)$ is the unique global solution to the homogeneous Cauchy problem

$$\begin{cases} \partial_t v - \operatorname{div}(A\nabla v) = 0 \\ v(0) = u_0 \end{cases}, \tag{HC}$$

with $\nabla v \in L^2(\mathbb{R}^{1+n}_+)$. Meanwhile, as $\nabla \mathcal{E}_{-\Delta}(u_0) \in L^2(\mathbb{R}^{1+n}_+)$, it also shows that $\widetilde{v} := \mathcal{R}^L_{1/2}((A - \mathbb{I})\nabla \mathcal{E}_{-\Delta}(u_0))$ is a global solution to the Cauchy problem

$$\begin{cases} \partial_t \widetilde{v} - \operatorname{div}(A\nabla \widetilde{v}) = \operatorname{div}((A - \mathbb{I})\nabla \mathcal{E}_{-\Delta}(u_0)) \\ \widetilde{v}(0) = 0 \end{cases}.$$

Notice that $\mathcal{E}_{-\Delta}(u_0) + \widetilde{v}$ is a global weak solution to the homogeneous Cauchy problem (HC) in the same class as $\mathcal{E}_L(u_0)$. Then by uniqueness, we deduce $\mathcal{E}_L(u_0) = \mathcal{E}_{-\Delta}(u_0) + \widetilde{v}$ as wanted for (6.9).

One can also verify (6.10) via an analogous argument. Details are left to the reader. This completes the proof. □

6.1.3 L^p-theory for the semigroup

The L^p-theory of the semigroup (e^{-tL}) is ruled by four critical numbers, which are introduced in [Aus07, Proposition 3.15] for $1 < p < \infty$, and later extended to $p > \frac{n}{n+1}$ in [AE23a, §6] to include Hardy spaces $H^p(\mathbb{R}^n)$. These numbers are

- $p_\pm(L) \in [\frac{n}{n+1}, \infty]$ such that $(p_-(L), p_+(L))$ is the largest open set (an interval) of exponents p for which the semigroup $(e^{-tL})_{t\geq 0}$ is uniformly bounded on L^p when $p > 1$ and on H^p when $p \leq 1$;
- $q_\pm(L) \in [\frac{n}{n+1}, \infty]$ such that $(q_-(L), q_+(L))$ is the largest open set (an interval) of exponents p for which the family $(t^{1/2}\nabla e^{-tL})_{t>0}$ is uniformly bounded on L^p when $p > 1$ and on H^p when $p \leq 1$.

It is known that

$$\begin{aligned} &p_-(L) = q_-(L) < \frac{2n}{n+2}, \\ &p_+(L) \geq \frac{nq_+(L)}{n - q_+(L)}, \quad q_+(L) > 2. \end{aligned}$$

Here, by convention, $p_+(L) = \infty$ if $q_+(L) \geq n$. The strict inequalities are best possible. Note that $\frac{nq_+(L)}{n-q_+(L)}$ is the Sobolev conjugate of $q_+(L)$ and $\frac{2n}{n+2}$ is the Hölder conjugate of the Sobolev conjugate of 2. These critical numbers are related by duality. Indeed, define

$$\boxed{p_-^\flat(L) := \max\{p_-(L), 1\}}.$$

Then we have the duality relation

$$p_+(L^*) = p_-^\flat(L)',$$

where p' denotes the Hölder conjugate of $p \in [1, \infty]$. For the negative Laplacian $L = -\Delta$, we know

$$\begin{cases} p_-(-\Delta) = q_-(-\Delta) = \frac{n}{n+1}, & p_-^\flat(-\Delta) = 1, \\ p_+(-\Delta) = q_+(-\Delta) = \infty. \end{cases}$$

Moreover, these critical numbers provide finer descriptions on the continuity and decay properties of the semigroup. Let us first recall the off-diagonal estimates of Davies–Gaffney type. We follow the notation $[p,q] := \frac{1}{p} - \frac{1}{q}$ for any $p, q \in (0, \infty]$, if there is no confusion with closed intervals.

Definition 6.8 (Off-diagonal estimates)**.** Let $1 \leq p \leq q \leq \infty$ and $M > 0$ be constants. We say that a family of bounded operators $(T_t)_{t>0}$ on $L^2(\mathbb{R}^n)$ has *(parabolic) $L^p - L^q$ off-diagonal estimates of order M*, if there is a constant $C > 0$ so that, for any Borel sets $E, F \subset \mathbb{R}^n$, $f \in L^2 \cap L^p$, and a.e. $t > 0$,

$$\|\mathbb{1}_E T_t \mathbb{1}_F f\|_q \leq C t^{-\frac{n}{2}[p,q]} \left(1 + \frac{\mathrm{dist}(E,F)^2}{t}\right)^{-M} \|\mathbb{1}_F f\|_p.$$

Proposition 6.9. (i) *Let $p_-^\flat(L) < p \leq q < p_+(L)$. The semigroup $(e^{-tL})_{t\geq 0}$ has $L^p - L^q$ off-diagonal estimates of any order M. Moreover, it is strongly continuous on L^p.*

(ii) *Let $p_-^\flat(L) < p \leq q < q_+(L)$. The family $(t^{1/2}\nabla e^{-tL})_{t>0}$ has $L^p - L^q$ off-diagonal estimates of any order M.*

(iii) *Let $p_-^\flat(L) < p \leq q < p_+(L)$. The family $(tLe^{-tL})_{t>0}$ has $L^p - L^q$ off-diagonal estimates of any order M.*

Proof. See [Aus07, Chapter 3]. □

6.2 Inhomogeneous Cauchy problems

We may now begin by studying the existence of weak solutions in weighted tent spaces of weak solutions to the *inhomogeneous Cauchy problem*

$$\begin{cases} \partial_t u - \mathrm{div}_x(A(x)\nabla_x u) = f, & (t,x) \in (0,\infty) \times \mathbb{R}^n, \\ u(0) = 0 \end{cases} \tag{IC}$$

6.2.1 Main results

The weak solution is constructed by the Duhamel's operator $\mathcal{L}_1^L$ defined in (6.6). Recall that it is initially defined from $L^2(\mathbb{R}^{1+n}_+)$ to $L^2_{\mathsf{loc}}((0,\infty);L^2)$ by the L^2-valued Bochner integrals

$$\mathcal{L}_1^L(f)(t) := \int_0^t e^{-(t-s)L} f(s)\,ds, \quad t>0.$$

Convention 6.10. Throughout this section, we abbreviate

$$\boxed{\mathcal{L}_1 := \mathcal{L}_1^L}.$$

Let us introduce the bound

$$p_L^\flat(\beta) := \frac{np_-^\flat(L)}{n+(2\beta+1)p_-^\flat(L)}. \tag{6.11}$$

Note that $p_L^\flat(\beta)$ agrees with $p_q(\beta)$ defined in (4.12) for $q=p_-^\flat(L)$ and $m=2$.

The following main theorem summarizes the properties of the extension. Recall that for any $(t,x)\in\mathbb{R}^{1+n}_+$, the (parabolic) Whitney cube $W(t,x)$ is defined by $W(t,x):=(t,2t)\times B(x,t^{1/2})$.

Theorem 6.11 (Extension of $\mathcal{L}_1$)**.** *Let $\beta>-1/2$ and $p_L^\flat(\beta)<p\le\infty$. Then $\mathcal{L}_1$ extends to a bounded operator from T^p_β to $T^p_{\beta+1}$, also denoted by $\mathcal{L}_1$, such that the following properties hold for any $f\in T^p_\beta$ and $u:=\mathcal{L}_1(f)$.*

(a) (Regularity) *u lies in $T^p_{\beta+1}$ and ∇u lies in $T^p_{\beta+1/2}$ with*

$$\|u\|_{T^p_{\beta+1}} \lesssim \|f\|_{T^p_\beta}, \quad \|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|f\|_{T^p_\beta}.$$

(b) *For any $\phi\in C_c^\infty(\overline{\mathbb{R}^{1+n}_+})$,*

$$\int_{\mathbb{R}^{1+n}_+} |u||\partial_t\phi| + \int_{\mathbb{R}^{1+n}_+} |A\nabla u||\nabla\phi| + \int_{\mathbb{R}^{1+n}_+} |f||\phi| \lesssim_\phi \|f\|_{T^p_\beta}, \tag{6.12}$$

and

$$-\int_{\mathbb{R}^{1+n}_+} u\overline{\partial_t\phi} + \int_{\mathbb{R}^{1+n}_+} (A\nabla u)\cdot\overline{\nabla\phi} = \int_{\mathbb{R}^{1+n}_+} f\overline{\phi}. \tag{6.13}$$

Therefore, u is a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u)=f$.

(c) (Maximal regularity) *$\partial_t u$ and $\operatorname{div}(A\nabla u)$ belong to T^p_β with*

$$\|\partial_t u\|_{T^p_\beta} + \|\operatorname{div}(A\nabla u)\|_{T^p_\beta} \lesssim \|f\|_{T^p_\beta}.$$

(d) (Whitney trace) *For a.e. $x\in\mathbb{R}^n$,*

$$\lim_{t\to 0}\left(\fint_{W(t,x)} |u(s,y)|^2 dsdy\right)^{1/2} = 0.$$

(e) (Continuity and distributional trace) $u \in C([0,\infty);\mathscr{S}')$ *with* $u(0)=0$. *As* $t \to 0$*, the convergence also occurs in*

$$\begin{cases} L^p & \text{if } p_L^\flat(\beta) < p \le 2 \\ E_\delta^q & \text{if } 2 < p \le \infty \end{cases},$$

where $q \in [p,\infty]$ *and* $\delta > 0$ *are arbitrary parameters.*

Consequently, u *is a global weak solution to* (IC) *with source term* f.

Statements (a) and (b) are proved in Section 6.2.3, and the others are proved in Section 6.2.5. Let us first make some remarks.

In (b), not only do we prove that $\mathcal{L}_1(f)$ is a global weak solution to the equation $\partial_t u - \operatorname{div}(A\nabla u) = f$, but we also get $\mathcal{L}_1(f)(0) = 0$ as the test functions are arbitrary at $t=0$. In turn, (d) and (e) explicitly show the null *boundary behavior* of $\mathcal{L}_1(f)(t)$ as $t \to 0$. As we shall see in Lemma 6.18, (d) in fact holds for any function $u \in T^p_{\beta+1}$.

Furthermore, (b) yields that the equality $\partial_t u - \operatorname{div}(A\nabla u) = f$ holds in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$, but (c) implies that it actually holds in T^p_β. In fact, $\partial_t u$ and $\operatorname{div}(A\nabla u)$ cannot lie in a better space simultaneously, otherwise the regularity of f could be raised. It is exactly the spirit of maximal regularity although the interpretation is no longer via semigroup theory.

6.2.2 L^2-theory

As we mentioned in Remark 6.6, the L^2-theory of (IC) is not covered by the energy class. Instead, our strategy is to exploit the operator theory itself to establish the L^2-theory for (IC) as *a priori* estimates. Precisely, we prove the following

Proposition 6.12 (L^2-theory of (IC))**.** *Let* $\beta > -1/2$ *and* $p_L^\flat(\beta) < p \le \infty$. *Let* f *be in* $L^2_c(\mathbb{R}^{1+n}_+)$.

(i) $\mathcal{L}_1(f)$ *lies in* $T^p_{\beta+1}$ *and* $\nabla\mathcal{L}_1(f)$ *lies in* $T^p_{\beta+1/2}$ *with*

$$\|\mathcal{L}_1(f)\|_{T^p_{\beta+1}} \lesssim \|f\|_{T^p_\beta}, \quad \|\nabla\mathcal{L}_1(f)\|_{T^p_{\beta+1/2}} \lesssim \|f\|_{T^p_\beta}.$$

(ii) *For any* $\phi \in C_c^\infty(\overline{\mathbb{R}^{1+n}_+})$,

$$\int_{\mathbb{R}^{1+n}_+} |\mathcal{L}_1(f)||\partial_t\phi| + \int_{\mathbb{R}^{1+n}_+} |A\nabla\mathcal{L}_1(f)||\nabla\phi| + \int_{\mathbb{R}^{1+n}_+} |f||\phi| \lesssim_\phi \|f\|_{T^p_\beta}, \tag{6.14}$$

and

$$-\int_{\mathbb{R}^{1+n}_+} \mathcal{L}_1(f)\overline{\partial_t\phi} + \int_{\mathbb{R}^{1+n}_+} (A\nabla\mathcal{L}_1(f))\cdot\overline{\nabla\phi} = \int_{\mathbb{R}^{1+n}_+} f\overline{\phi}. \tag{6.15}$$

In particular, u *is a global weak solution to* $\partial_t u - \operatorname{div}(A\nabla u) = f$.

Define the operator $\mathcal{L}_{1/2}$ from $L^2(\mathbb{R}^{1+n}_+)$ to $L^1_{\mathsf{loc}}((0,\infty);L^2)$ by the L^2-valued Bochner integrals

$$\mathcal{L}_{1/2}(f)(t) := \int_0^t \nabla e^{-(t-s)L} f(s)ds, \quad t>0. \tag{6.16}$$

The first lemma asserts that both $\mathcal{L}_1$ and $\mathcal{L}_{1/2}$ are included in the framework of singular integral operators on tent spaces developed in Chapter 4.

Lemma 6.13. *It holds that*

$$\begin{cases} \mathcal{L}_1 \in \mathrm{SIO}^{1+}_{2,q,\infty} & \textit{if } p^\flat_-(L) < q < p_+(L), \\ \mathcal{L}_{1/2} \in \mathrm{SIO}^{\frac{1}{2}+}_{2,q,\infty} & \textit{if } p^\flat_-(L) < q < q_+(L). \end{cases} \tag{6.17}$$

Moreover, for any $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$,

$$\mathcal{L}_{1/2}(f) = \nabla \mathcal{L}_1(f) \quad \textit{in } \mathscr{D}'(\mathbb{R}^{1+n}_+). \tag{6.18}$$

In particular, (6.15) holds if and only if for all $\phi \in C^\infty_{\mathsf{c}}(\overline{\mathbb{R}^{1+n}_+})$,

$$-\int_{\mathbb{R}^{1+n}_+} \mathcal{L}_1(f)\, \overline{\partial_t \phi} + \int_{\mathbb{R}^{1+n}_+} A\mathcal{L}_{1/2}(f) \cdot \overline{\nabla \phi} = \int_{\mathbb{R}^{1+n}_+} f \overline{\phi}. \tag{6.19}$$

Proof. Let us first prove (6.17). It suffices to show the first, and the second follows similarly. When $p^\flat_-(L) < q \le r < p_+(L)$, the family $(e^{-(t-s)L})_{t>s}$ has $L^q - L^r$ off-diagonal decay of type $(1,2,M)$ for any $M>0$. Thus, for $p^\flat_-(L) < q < p_+(L)$, the function $(t,s) \mapsto \mathbb{1}_{\{t>s\}}(t,s)e^{-(t-s)L}$ belongs to $\mathrm{SK}^1_{2,q,\infty}$. In consequence, Lemma 4.6 implies $\mathcal{L}_1 \in \mathrm{SIO}^{1+}_{2,q,\infty}$.

Next, we prove (6.18). Let $\psi \in C^\infty_{\mathsf{c}}(\mathbb{R}^{1+n}_+;\mathbb{C}^n)$. Fubini's theorem yields

$$\begin{aligned} \int_{\mathbb{R}^{1+n}_+} \mathcal{L}_{1/2}(f) \cdot \overline{\psi} &= \int_0^\infty dt \int_0^t \langle \nabla e^{-(t-s)L} f(s), \psi(t) \rangle_{L^2(\mathbb{R}^n;\mathbb{C}^n)} ds \\ &= -\int_0^\infty dt \int_0^t \langle e^{-(t-s)L} f(s), \operatorname{div} \psi(t) \rangle_{L^2(\mathbb{R}^n)} ds \\ &= -\int_{\mathbb{R}^{1+n}_+} \mathcal{L}_1(f) \operatorname{div} \overline{\psi}. \end{aligned}$$

Finally, to show (6.19), we only need to observe that for any $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$, (6.18) says the identity $\nabla \mathcal{L}_1(f) = \mathcal{L}_{1/2}(f)$ holds in $L^1_{\mathsf{loc}}((0,\infty);L^2)$, so the equivalence clearly follows. This completes the proof. $\square$

We also need an embedding lemma.

Lemma 6.14. *Let $m \in \mathbb{N}$ and $\beta > -1/2$. Then, $L^\infty_{\mathsf{c}}(\overline{\mathbb{R}^{1+n}_+})$ embeds into $T^{p;m}_{-\beta}$ and $T^{\infty;m}_{-\beta;(\sigma)}$ for $p \in (0,\infty)$ and $0 \le \sigma \le \frac{m}{n}(\beta + 1/2)$.*

Proof. Let ϕ be an $L^\infty_{\mathsf{c}}(\overline{\mathbb{R}^{1+n}_+})$-function supported in $[0,R] \times B(0,R^{1/m})$ for some $R > 0$. Then $\mathcal{A}^{(1)}_{-\beta;m}(\phi)$ is supported on $B(0,2R^{1/m})$, where $\mathcal{A}^{(1)}_{-\beta;m}$ is defined in (3.1). Moreover, for any $x \in B(0,2R^{1/m})$,

$$\mathcal{A}^{(1)}_{-\beta;m}(\phi)(x)^2 = \int_0^R dt \fint_{B(x,t^{1/m})} |t^\beta \phi(t,y)|^2 dy \le \|\phi\|_\infty^2 \int_0^R t^{2\beta} dt \lesssim \|\phi\|_\infty^2$$

as $\beta > -1/2$. Thus, for any $p \in (0,\infty)$,

$$\|\phi\|_{T^{p;m}_{-\beta}} \eqsim \|\mathcal{A}^{(1)}_{-\beta;m}(\phi)\|_p \lesssim_{R,\beta} \|\phi\|_\infty.$$

Similarly, for any ball $B = B(x_0,r^{1/m}) \subset \mathbb{R}^n$, we get

$$\begin{aligned}
&\frac{1}{|B|^\sigma}\left(\int_0^r \mathbb{1}_{[0,R]}(t) dt \fint_B |t^\beta \phi(t,y)|^2 dy\right)^{1/2} \\
&\quad \lesssim r^{-\frac{n}{m}\sigma}\left(\int_0^{\min\{r,R\}} t^{2\beta} dt\right)^{1/2} \|\phi\|_\infty \lesssim R^{\beta+1/2-\frac{n}{m}\sigma}\|\phi\|_\infty.
\end{aligned}$$

We hence conclude by taking the supremum over all balls B, since the controlling constant is independent of B. ☐

We now prove Proposition 6.12.

Proof of Proposition 6.12. First consider (i). Since $L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$ is contained in T^p_β, applying Proposition 4.12 and Corollary 4.14, we get (i) as

$$\|\mathcal{L}_1(f)\|_{T^p_{\beta+1}} \lesssim \|f\|_{T^p_\beta}, \quad \|\nabla \mathcal{L}_1(f)\|_{T^p_{\beta+1/2}} = \|\mathcal{L}_{1/2}(f)\|_{T^p_{\beta+1/2}} \lesssim \|f\|_{T^p_\beta}.$$

The conditions there are verified in Lemma 6.13.

Then consider (ii). Fix $\phi \in C^\infty_{\mathsf{c}}(\overline{\mathbb{R}^{1+n}_+})$. For (6.14), Lemma 6.14 yields $\phi \in (T^p_\beta)'$, $\nabla\phi \in (T^p_{\beta+1/2})'$, and $\partial_t \phi \in (T^p_{\beta+1})'$. Using duality of tent spaces and (i), we get

$$\text{LHS } (6.14) \lesssim_\phi \|\mathcal{L}_1(f)\|_{T^p_{\beta+1}} + \|\nabla \mathcal{L}_1(f)\|_{T^p_{\beta+1/2}} + \|f\|_{T^p_\beta} \lesssim \|f\|_{T^p_\beta}.$$

This proves (6.14). For (6.15), the proof is divided into two cases.

Case 1: $f \in L^2_{\mathsf{c}}((0,\infty);D(L))$ Thanks to the analyticity of (e^{-tL}) on L^2, we get $\mathcal{L}_1(f) \in C([0,\infty);L^2)$ with $\mathcal{L}_1(f)(0) = 0$, and $\mathcal{L}_1(f)(t) \in D(L)$ for all $t > 0$. Thus, we infer $L\mathcal{L}_1(f) \in L^2(\mathbb{R}^{1+n}_+)$, so

$$\partial_t \mathcal{L}_1(f) = f + \operatorname{div}(A\nabla \mathcal{L}_1(f)) = f - L\mathcal{L}_1(f) \in L^2(\mathbb{R}^{1+n}_+). \tag{6.20}$$

Then (6.15) holds by testing $\phi \in C^\infty_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$ against (6.20) and using integration by parts.

Case 2: $f \in L^2_c(\mathbb{R}^{1+n}_+)$ Lemma 6.13 says that it is equivalent to prove (6.19). Lemma 6.14 implies $\phi \in L^2(\mathbb{R}^{1+n}_+)$, $\nabla\phi \in L^2_{-1/2}(\mathbb{R}^{1+n}_+)$, and $\partial_t\phi \in L^2_{-1}(\mathbb{R}^{1+n}_+)$. Ensured by the fact that $D(L)$ is dense in L^2, a standard density argument verifies (6.19), also using Case 1 and continuity of $\mathcal{L}_1 : L^2(\mathbb{R}^{1+n}_+) \to L^2_1(\mathbb{R}^{1+n}_+)$ and $\mathcal{L}_{1/2} : L^2(\mathbb{R}^{1+n}_+) \to L^2_{1/2}(\mathbb{R}^{1+n}_+)$. This completes the proof. □

6.2.3 Existence

In this section, we construct the extension of $\mathcal{L}_1$, and prove Theorem 6.11 (a) and (b).

Lemma 6.15. *Let $\beta > -1/2$ and $p^\flat_L(\beta) < p \le \infty$. Then, for $\kappa = 1$ and $\kappa = 1/2$, $\mathcal{L}_\kappa$ extends to bounded operators from T^p_β to $T^p_{\beta+\kappa}$. Furthermore, (6.18) holds for the extensions applied to $f \in T^p_\beta$.*

Proof. Using (6.17), the extension follows by applying Proposition 4.12 and Corollary 4.14. In the following, the extension of $\mathcal{L}_\kappa$ to tent spaces is still denoted by $\mathcal{L}_\kappa$. We next divide the proof of (6.18) in two cases.

Case 1: $p^\flat_L(\beta) < p < \infty$ Lemma 6.13 says that it holds when $f \in L^2_c(\mathbb{R}^{1+n}_+)$. A standard density argument extends it to $f \in T^p_\beta$, ensured by three ingredients: the density of $L^2_c(\mathbb{R}^{1+n}_+)$ in T^p_β; the continuity of $\mathcal{L}_\kappa$ from T^p_β to $T^p_{\beta+\kappa}$ for $\kappa = 1, 1/2$; and the fact that any $C^\infty_c(\mathbb{R}^{1+n}_+)$-function belongs to $(T^p_{\beta+1})' \cap (T^p_{\beta+1/2})'$, see Lemma 6.14.

Case 2: $p = \infty$ Let $\mathcal{L}^*_\kappa$ be the adjoint of $\mathcal{L}_\kappa$ with respect to $L^2(\mathbb{R}^{1+n}_+)$-duality for $\kappa = 1, 1/2$. Lemma 3.4 shows $L^2_c(\mathbb{R}^{1+n}_+)$ is weak* dense in T^∞_β, so the desired distribution equality follows by the continuity of $\mathcal{L}^*_\kappa : T^1_{-\beta-\kappa} \to T^1_{-\beta}$ for $\kappa = 1, 1/2$, and the fact that $C^\infty_c(\mathbb{R}^{1+n}_+)$ is contained in $T^1_{-\beta-1} \cap T^1_{-\beta-1/2}$. This completes the proof. □

We can now prove Theorem 6.11 (a) and (b).

Proof of Theorem 6.11 (a) and (b). The extension is given by Lemma 6.15. It also implies that if $u = \mathcal{L}_1(f)$ with $f \in T^p_\beta$, then $\nabla u = \mathcal{L}_{1/2}(f)$, so (a) follows by the boundedness of $\mathcal{L}_1$ and $\mathcal{L}_{1/2}$.

Then consider (b). Since all the tent spaces embed into $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$, we get that both of u and ∇u belong to $L^2_{\mathrm{loc}}(\mathbb{R}^{1+n}_+)$, so $u \in L^2_{\mathrm{loc}}((0,\infty); W^{1,2}_{\mathrm{loc}})$. The inequality (6.12) is also clear by Lemma 6.14 and (a), so it only remains to prove (6.13). Here, we can follow the proof of Proposition 6.12 (ii), that is to say,

- Interpreting (6.13) as (6.19) using duality of tent spaces, since $\nabla\mathcal{L}_1(f)$ agrees with $\mathcal{L}_{1/2}(f)$ in $T^p_{\beta+1/2}$ and $\nabla\phi \in (T^p_{\beta+1/2})'$;

- Showing (6.19) by density with a sequence in $L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$ that approximates f in T^p_β (or weak* if $p=\infty$, see Case 2 of Lemma 6.15).

All the ingredients have been provided in Lemma 6.14 and 6.15 and details are left to the reader. □

6.2.4 Maximal regularity

Define the *maximal regularity operator* $\mathcal{L}_0$ on $L^2((0,\infty);D(L))$ by

$$\mathcal{L}_0(f)(t) := \int_0^t Le^{-(t-s)L} f(s)ds, \quad t>0. \tag{6.21}$$

The integral makes sense as an $L^2(\mathbb{R}^n)$-valued Bochner integral. Recall that the celebrated de Simon's theorem [dS64] asserts $\mathcal{L}_0$ can be extended by density to a bounded operator on $L^2(\mathbb{R}^{1+n}_+)$. Denote by $\tilde{\mathcal{L}}_0$ this extension.

Lemma 6.16. *$\tilde{\mathcal{L}}_0$ belongs to $\mathrm{SIO}^{0+}_{2,q,\infty}$ for $p_-(L)<q<p_+(L)$.*

Proof. The boundedness of $\tilde{\mathcal{L}}_0$ on $L^2(\mathbb{R}^{1+n}_+)$ is clear by construction. We also infer from Proposition 6.9 that the function $(t,s)\mapsto \mathbb{1}_{\{t>s\}}(t,s)Le^{-(t-s)L}$ belongs to $\mathrm{SK}^0_{2,q,\infty}$ for $p_-(L)<q<p_+(L)$. Thus, Lemma 4.8 yields for any $f\in L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$,

$$(t,x)\mapsto \int_0^t (Le^{-(t-s)L}f)(x)ds$$

defines a function in $L^2_{\mathsf{loc}}(\overline{\mathbb{R}^{1+n}_+}\setminus \pi(f))$. It only remains to prove the representation (4.10), or equivalently, for any $f\in L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$ and $g\in C^\infty_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$ with $\pi(g)\cap\pi(f)=\varnothing$,

$$\left\langle \tilde{\mathcal{L}}_0(f),g\right\rangle_{L^2(\mathbb{R}^{1+n}_+)} = \int_0^\infty \int_{\mathbb{R}^n} \left(\int_0^t (Le^{-(t-s)L}f(s))(x)ds\right)\overline{g}(t,x)dtdx. \tag{6.22}$$

To prove (6.22), we proceed by putting the following two observations together. Fix $g\in C^\infty_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$.

The first observation is that for any $f\in L^2((0,\infty);D(L))$,

$$\langle \mathcal{L}_0(f),g\rangle_{L^2(\mathbb{R}^{1+n}_+)} = \left\langle A\mathcal{L}_{1/2}(f),\nabla g\right\rangle_{L^2(\mathbb{R}^{1+n}_+)}. \tag{6.23}$$

Indeed, since $\mathcal{L}_{1/2}(f)$ is given by an $L^2(\mathbb{R}^n)$-valued Bochner integral, all the following integrals converge absolutely, so Fubini's theorem ensures

$$\begin{aligned}\langle \mathcal{L}_0(f),g\rangle_{L^2(\mathbb{R}^{1+n}_+)} &= \int_0^\infty\int_0^t \left\langle Le^{-(t-s)L}f(s),g(t)\right\rangle_{L^2(\mathbb{R}^n)} dsdt \\ &= \int_0^\infty\int_0^t \left\langle A\nabla e^{-(t-s)L}f(s),\nabla g(t)\right\rangle_{L^2(\mathbb{R}^n)} dsdt \\ &= \left\langle A\mathcal{L}_{1/2}(f),\nabla g\right\rangle_{L^2(\mathbb{R}^{1+n}_+)}.\end{aligned}$$

The second observation is that when f lies in $L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$ with $\pi(f)$ disjoint with $\pi(g)$, we have the identity

$$\begin{aligned}
&\langle A\mathcal{L}_{1/2}(f), \nabla g\rangle_{L^2(\mathbb{R}^{1+n}_+)} \\
&= \int_0^\infty \int_0^t \int_{\mathbb{R}^n} (A\nabla e^{-(t-s)L} f(s))(x) \cdot \overline{\nabla g}(t,x) dx ds dt \\
&= -\int_0^\infty \int_0^t \int_{\mathbb{R}^n} \operatorname{div}(A\nabla e^{-(t-s)L} f(s))(x) \overline{g}(t,x) dx ds dt =: I
\end{aligned} \tag{6.24}$$

Indeed, all the integrals converge absolutely as for fixed (t,s), both $x \mapsto (A\nabla e^{-(t-s)L} f(s))(x)$ and $x \mapsto \operatorname{div}(A\nabla e^{-(t-s)L} f(s))(x)$ are integrable on the support of $g(t)$, using Lemma 4.6 with $\kappa = 1/2$ and Lemma 4.8 with $\kappa = 0$. Thus, the second equality follows by integration by parts for the divergence.

We now prove (6.22) as follows. By a limiting argument from the first observation, we have that for any $f \in L^2(\mathbb{R}^{1+n}_+)$, (6.23) becomes

$$\left\langle \tilde{\mathcal{L}}_0(f), g \right\rangle_{L^2(\mathbb{R}^{1+n}_+)} = \langle A\mathcal{L}_{1/2}(f), \nabla g\rangle_{L^2(\mathbb{R}^{1+n}_+)} \tag{6.25}$$

Indeed, the argument is similar to Lemma 6.13, but using the continuity of $\mathcal{L}_{1/2}$ and $\tilde{\mathcal{L}}_0$. We hence leave details to the reader.

Next, for any $f \in L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$ with $\pi(f)$ disjoint with $\pi(g)$, (6.25) and (6.24) yield

$$\left\langle \tilde{\mathcal{L}}_0(f), g \right\rangle_{L^2(\mathbb{R}^{1+n}_+)} = I.$$

So it only remains to observe that for a.e. $s < t$ and $x \in \mathbb{R}^n$,

$$\operatorname{div}(A\nabla e^{-(t-s)L} f(s))(x) = (Le^{-(t-s)L} f(s))(x)$$

by definition of L, as $f(s)$ belongs to $L^2(\mathbb{R}^n)$. $\square$

We then consider a further extension of $\tilde{\mathcal{L}}_0$ to tent spaces.

Proposition 6.17. *Let $\beta > -1/2$ and $p_L^\flat(\beta) < p \le \infty$. Then, $\tilde{\mathcal{L}}_0$ can be extended to a bounded operator on T^p_β.*

Proof. Lemma 6.16 says $\tilde{\mathcal{L}}_0$ lies in $\mathrm{SIO}^{0+}_{2,q,\infty}$ when $p_-(L) < q < p_+(L)$, and [AA11a, Theorem 1.3] shows $\tilde{\mathcal{L}}_0$ is bounded on $L^2_\beta(\mathbb{R}^{1+n}_+)$ for any $\beta > -1/2$. We hence conclude by invoking Proposition 4.12 and Corollary 4.14. $\square$

The extension of $\tilde{\mathcal{L}}_0$ to T^p_β is still denoted by $\tilde{\mathcal{L}}_0$ in the sequel. Write $\mathcal{L}_1$ and $\mathcal{L}_{1/2}$ for their extensions to T^p_β. Let us prove Theorem 6.11 (c).

Proof of Theorem 6.11 (c). Let f be in T^p_β and $u = \mathcal{L}_1(f)$. It suffices to identify $-\operatorname{div}(A\nabla u)$ with $\tilde{\mathcal{L}}_0(f)$ in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$. Indeed, if it holds, then Proposition 6.17 yields

$$\|\operatorname{div}(A\nabla u)\|_{T^p_\beta} = \|\tilde{\mathcal{L}}_0(f)\|_{T^p_\beta} \lesssim \|f\|_{T^p_\beta}.$$

Moreover, (b) implies $\partial_t u = \operatorname{div}(A\nabla u) + f$ in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$. Thus, it follows directly that $\partial_t u$ lies in T^p_β with

$$\|\partial_t u\|_{T^p_\beta} \lesssim \|\operatorname{div}(A\nabla u)\|_{T^p_\beta} + \|f\|_{T^p_\beta} \lesssim \|f\|_{T^p_\beta}.$$

To prove the identification, we first observe that

$$\tilde{\mathcal{L}}_0(f) = -\operatorname{div}(A\mathcal{L}_{1/2}(f)) \quad \text{in } \mathscr{D}'(\mathbb{R}^{1+n}_+). \tag{6.26}$$

In fact, it follows from (6.25) with a limiting argument applied to a sequence in $T^p_\beta \cap L^2(\mathbb{R}^{1+n}_+)$ approximating f (weak* if $p=\infty$, see Case 2 of Lemma 6.15). The second observation is that

$$-\operatorname{div}(A\mathcal{L}_{1/2}(f)) = -\operatorname{div}(A\nabla\mathcal{L}_1(f)) \quad \text{in } \mathscr{D}'(\mathbb{R}^{1+n}_+). \tag{6.27}$$

Indeed, Lemma 6.15 says $\mathcal{L}_{1/2}(f) = \nabla\mathcal{L}_1(f)$ in $T^p_{\beta+1/2}$, hence in $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$, so (6.27) follows by multiplying by A and applying the divergence. Combining (6.26) and (6.27) yields $\tilde{\mathcal{L}}_0(f) = -\operatorname{div}(A\nabla u)$ as desired. □

6.2.5 Continuity and traces

In this section, we prove Theorem 6.11 (d) and (e). The first result is valid for arbitrary function in $T^p_{\beta+1}$.

Lemma 6.18. *Let $0<p\le\infty$ and $\beta>-1/2$. Let u be in $T^p_{\beta+1}$. Then, for a.e. $x\in\mathbb{R}^n$,*

$$\lim_{t\to 0}\left(\fint_{W(t,x)} |u(s,y)|^2 ds dy\right)^{1/2} = 0.$$

Proof. First consider $0<p<\infty$. For any $t>0$ and $x\in\mathbb{R}^n$, we have

$$\fint_{W(t,x)} |u(s,y)|^2 dsdy \lesssim \fint_t^{2t} ds \fint_{B(x,s^{1/2})} |u(s,y)|^2 dy \lesssim t^{2\beta+1}\left(\mathcal{A}^{(1)}_{\beta+1;2}(u)(x)\right)^2,$$

where $\mathcal{A}^{(1)}_{\beta+1;2}$ is defined in (3.1). Since $\|\mathcal{A}^{(1)}_{\beta+1;2}(u)\|_p \eqsim \|u\|_{T^p_{\beta+1}} < \infty$, $\mathcal{A}^{(1)}_{\beta+1}(u)(x)$ is finite for a.e. $x\in\mathbb{R}^n$, so the right-hand side tends to 0 when $t\to 0$ as $\beta>-1/2$.

Then for $p=\infty$, note that $W(t,x)\subset(0,2t)\times B(x,(2t)^{1/2})$, so

$$\fint_{W(t,x)} |u(s,y)|^2 dsdy \lesssim t^{2\beta+1}\int_t^{2t} ds \fint_{B(x,(2t)^{1/2})} |s^{-(\beta+1)}u(s,y)|^2 dy.$$

Taking esssup on $x\in\mathbb{R}^n$, we get

$$\operatorname*{esssup}_{x\in\mathbb{R}^n}\fint_{W(t,x)} |u(s,y)|^2 dsdy \lesssim t^{2\beta+1}\|u\|^2_{T^\infty_{\beta+1}}.$$

This completes the proof. □

Lemma 6.19. *For any $p \in [2,\infty]$ and $\beta \geq -1$, $T^p_{\beta+1}$ is contained in $L^2_{\mathsf{loc}}(\overline{\mathbb{R}^{1+n}_+})$. More precisely, for ball $B \subset \mathbb{R}^n$ and $0 < T < r(B)^2$,*

$$\left(\int_0^T \int_B |u(t,y)|^2 dtdy \right)^{1/2} \lesssim_p T^{\beta+1} |B|^{[2,p]} \|u\|_{T^p_{\beta+1}}. \tag{6.28}$$

Proof. By definition, it is trivial if $p = \infty$. We thus assume $2 \leq p < \infty$. For any $\alpha > 0$ and ball $B \subset \mathbb{R}^n$, define the *tent* on B of *aperture* α (and *homogeneity* $m = 2$) as

$$T_\alpha(B) := \{(t,y) \in \mathbb{R}^{1+n}_+ : 0 < \alpha t^{1/2} \leq \operatorname{dist}(y, B^c)\}.$$

Observe that $(0,T) \times B \subset T_\alpha(2B)$ with $\alpha := T^{-1/2} r(B) > 1$. Moreover, for any $(t,y) \in T_\alpha(2B)$, since $\alpha t^{1/2} < r(B) < \operatorname{dist}(y,(2B)^c)$, we know that $B(y, \alpha t^{1/2}) \subset 2B$. In other words,

$$\int_{2B} \mathbb{1}_{B(0,1)} \left(\frac{x-y}{\alpha t^{1/2}} \right) dx \eqsim (\alpha t^{1/2})^n.$$

With the help of these two observations, we get

$$\begin{aligned}
\|u\|^2_{L^2((0,T)\times B)} &\lesssim \alpha^{-n} \int_{T_\alpha(2B)} |u(t,y)|^2 dt \frac{dy}{t^{n/2}} \int_{2B} \mathbb{1}_{B(0,1)} \left(\frac{x-y}{\alpha t^{1/2}} \right) dx \\
&\leq \int_{2B} dx \fint_0^T \int_{B(x,\alpha t^{1/2})} |u(t,y)|^2 dt \frac{dy}{t^{n/2}} \\
&\leq T^{2(\beta+1)} \int_{2B} \left(\mathcal{A}^{(\alpha)}_{\beta+1;2}(u)(x) \right)^2 dx \\
&\lesssim_p T^{2(\beta+1)} |B|^{2[2,p]} \|\mathcal{A}^{(\alpha)}_{\beta+1;2}(u)\|^2_p \lesssim T^{2(\beta+1)} |B|^{2[2,p]} \|u\|^2_{T^p_{\beta+1}}.
\end{aligned}$$

The last inequality follows from the change of aperture, cf. Lemma 3.12. □

Corollary 6.20. *Let $2 \leq p \leq \infty$ and $\beta \geq -1$. Let $B \subset \mathbb{R}^n$ be a ball and $0 < T < r(B)^2$. For any $u \in T^p_{\beta+1}$,*

$$\fint_0^T \|u(t)\|_{L^2(B)} dt \lesssim_\beta T^{\beta+1/2} |B|^{[2,p]} \|u\|_{T^p_{\beta+1}}. \tag{6.29}$$

In particular, if $\beta > -1/2$, $u(t)$ tends to 0 in a Cesàro mean in $L^2_{\mathsf{loc}}(\mathbb{R}^n)$ as $t \to 0$.

Proof. It suffices to prove (6.29), which is a direct consequence of Lemma 6.19 by Jensen's inequality as

$$\fint_0^T \|u(t)\|_{L^2(B)} dt \leq \left(\fint_0^T \|u(t)\|^2_{L^2(B)} dt \right)^{1/2} \lesssim T^{\beta+1/2} |B|^{[2,p]} \|u\|_{T^p_{\beta+1}}.$$

This completes the proof. □

Corollary 6.21. *Let $2 \le p \le \infty$ and $\beta \ge -1$. Let u be in $T^p_{\beta+1}$ with $\partial_t u \in T^p_\beta$. Then, for any $\delta > 0$ and $t \in (0,\delta)$,*

$$\|u(t)\|_{E^\infty_\delta} \lesssim \delta^{-\frac{n}{2p}} t^{\beta+1/2} \big(\|u\|_{T^p_{\beta+1}} + \|\partial_t u\|_{T^p_\beta} \big). \tag{6.30}$$

Proof. Fix a ball $B \subset \mathbb{R}^n$. Since all the tent spaces embed into $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$, both of u and $\partial_t u$ belong to $L^2((T/8, 2T) \times B)$ for any $T > 0$. Thus, for any $T' \in [T/4, T/2]$, we have

$$u(T) = u(T') + \int_{T'}^T \partial_t u(\tau) d\tau$$

valued in $L^2(B)$. Taking average with respect to T', we get

$$u(T) = \fint_{T/4}^{T/2} u(T') dT' + \fint_{T/4}^{T/2} dT' \int_{T'}^T \partial_t u(\tau) d\tau.$$

Since $T/4 \le T' \le \min\{\tau, T/2\}$, Fubini's theorem yields

$$\|u(T)\|_{L^2(B)} \lesssim \fint_{T/4}^{T/2} \|u(T')\|_{L^2(B)} dT' + \fint_{T/4}^{T} \tau \|\partial_t u(\tau)\|_{L^2(B)} d\tau.$$

Note that both of u and $\tau \partial_t u(\tau)$ lie in $T^p_{\beta+1}$. Using Corollary 6.20, we deduce

$$\|u(t)\|_{E^\infty_\delta} \eqsim \delta^{-\frac{n}{2}} \sup_B \|u(t)\|_{L^2(B)} \lesssim \delta^{-\frac{n}{2p}} t^{\beta+\frac{1}{2}} \big(\|u\|_{T^p_{\beta+1}} + \|\partial_t u\|_{T^p_\beta} \big),$$

where B describes all balls in $\mathbb{R}^n$ with $r(B) = \delta^{1/2}$. $\square$

We may finish the proof of Theorem 6.11.

Proof of Theorem 6.11 (d) and (e). Statement (d) is clear by Lemma 6.18. Let us focus on (e). First we claim that

$$\|u(t)\|_{E^p_{t/16}} \lesssim t^{\beta+1/2} (\|u\|_{T^p_{\beta+1}} + \|f\|_{T^p_\beta}). \tag{6.31}$$

Indeed, It follows by applying Caccioppoli's inequality (cf. Lemma 6.3) to the average of $u(t)$ on $B(x, \frac{\sqrt{t}}{4})$ as

$$\begin{aligned} \|u(t)\|^p_{E^p_{t/16}} &\lesssim \int_{\mathbb{R}^n} \left(\frac{1}{t} \int_{t/2}^t \fint_{B(x,\frac{\sqrt{t}}{2})} |u|^2 \right)^{p/2} dx + \int_{\mathbb{R}^n} \left(t \int_{t/2}^t \fint_{B(x,\frac{\sqrt{t}}{2})} |f|^2 \right)^{p/2} dx \\ &\lesssim t^{(\beta+1/2)p} (\|u\|^p_{T^p_{\beta+1}} + \|f\|^p_{T^p_\beta}). \end{aligned}$$

This proves the claim (6.31). Then the discussion is divided into two cases.

Case 1: $2 < p \le \infty$ Fix $\delta > 0$ and $t \in (0,\delta)$. Applying (6.31) and the change of aperture for slice spaces (cf. (3.2)) gives

$$\|u(t)\|_{E^p_\delta} \lesssim \|u(t)\|_{E^p_{t/16}} \lesssim t^{\beta+1/2}(\|u\|_{T^p_{\beta+1}} + \|f\|_{T^p_\beta}).$$

Moreover, (6.30) also holds for u by Corollary 6.21 as $u \in T^p_{\beta+1}$ (by (a)) and $\partial_t u \in T^p_\beta$ (by (c)). Interpolation of slice spaces hence implies $u(t)$ tends to 0 as $t \to 0$ in E^q_δ for any $q \in [p,\infty]$.

Note that E^q_δ embeds into $\mathscr{S}'$, so in particular, we have $u(t)$ tends to 0 in $\mathscr{S}'$ as $t \to 0$.

It only remains to show $u \in C((0,\infty);\mathscr{S}')$. Fix $t > 0$. For convenience, we prove that $u(s)$ converges to $u(t)$ in $\mathscr{S}'$ as $s \to t+$. The other side follows similarly. Fix $\phi \in \mathscr{S}$. For $s > t$, we infer from Lemma 5.23 (i) that the function $(\tau,y) \mapsto \mathbb{1}_{(t,s)}(\tau)\phi(y)$ lies in $T^{p'}_{-\beta}$, and there exists $N > 0$ so that

$$\|\mathbb{1}_{(t,s)}(\tau)\phi\|_{T^{p'}_{-\beta}} \lesssim (s^{2\beta+1} - t^{2\beta+1})^{1/2}\mathcal{P}_N(\phi).$$

Since $\partial_t u \in T^p_\beta$ (see (c)), by using duality of tent spaces, we obtain

$$\begin{aligned}|\langle u(s) - u(t), \phi\rangle| &\le \int_t^s \int_{\mathbb{R}^n} |\partial_t u(\tau,y)||\phi(y)| d\tau dy \\ &\lesssim \|\partial_t u\|_{T^p_\beta} \|\mathbb{1}_{(t,s)}(\tau)\phi\|_{T^{p'}_{-\beta}} \lesssim_t \|f\|_{T^p_\beta}\mathcal{P}_N(\phi)(s^{2\beta+1} - t^{2\beta+1})^{1/2},\end{aligned}$$

which tends to 0 as $s \to t+$. This concludes the case.

Case 2: $\max\{p^\flat_L(\beta), 1\} < p \le 2$ Observe that $E^p_{t/16}$ embeds into $L^p(\mathbb{R}^n)$ by Hölder's inequality, so (6.31) implies $u(t)$ tends to 0 as $t \to 0$ in L^p, and hence in $\mathscr{S}'$. Moreover, the same arguments as in Case 1 shows $u \in C((0,\infty);\mathscr{S}')$. This case is hence done.

Case 3: $p^\flat_L(\beta) < p \le 1$ Using (6.31) and Hölder's inequality, we can also get $u(t)$ tends to 0 as $t \to 0$ in L^p. However, for $p < 1$, L^p does not embed into $\mathscr{S}'$, so we need to prove $u \in C([0,\infty);\mathscr{S}')$ with $u(0) = 0$.

We first show $u \in C((0,\infty);\mathscr{S}')$. Let $t > 0$ and $\phi \in \mathscr{S}$. As seen in Case 1, we only need to show $u(s)$ converges to $u(t)$ as $s \to t+$. For $s > t$, again, Lemma 5.23 (i) implies there exists $N > 0$ so that

$$\|\mathbb{1}_{(t,s)}(\tau)\phi\|_{T^\infty_{-\beta,([p,1])}} \lesssim (s^{2\beta+1} - t^{2\beta+1})^{1/2}\mathcal{P}_N(\phi).$$

As $\partial_t u \in T^p_\beta$, by duality of tent spaces, we have

$$|\langle u(s) - u(t), \phi\rangle| \lesssim_t \|\partial_t u\|_{T^p_\beta}\mathcal{P}_N(\phi)(s^{2\beta+1} - t^{2\beta+1})^{1/2},$$

which tends to 0 as $s \to t+$.

To prove the continuity at $t = 0$, we notice that as $p > p_L^\flat(\beta) = \frac{np_-^\flat(L)}{n+(2\beta+1)p_-^\flat(L)}$, we have

$$[p, 1] = \frac{1}{p} - 1 \leq \frac{1}{p} - \frac{1}{p_-^\flat(L)} < \frac{2\beta + 1}{n}.$$

In the first inequality, we use the fact that $p_-^\flat(L) \geq 1$. Thus, Lemma 5.23 (ii) says

$$\|\mathbb{1}_{(0,t)}(\tau)\phi\|_{T^\infty_{-\beta,([p,1])}} \lesssim t^{\beta+\frac{1}{2}-\frac{n}{2}[p,1]}\mathcal{P}_0(\phi).$$

Therefore, by duality of tent spaces, we obtain

$$|\langle u(t), \phi\rangle| \lesssim \|\partial_t u\|_{T^p_\beta} t^{\beta+\frac{1}{2}-\frac{n}{2}[p,1]}\mathcal{P}_0(\phi),$$

which tends to 0 as $t \to 0$, since $\beta + \frac{1}{2} - \frac{n}{2}[p, 1] > 0$.

This completes the proof. □

6.3 Lions' equation

This section is concerned with existence of weak solutions to the *Lions equation*

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F \\ u(0) = 0 \end{cases}. \tag{L}$$

The weak solutions are constructed by the Lions operator $\mathcal{R}^L_{1/2}$ defined in (6.7). Recall that $\mathcal{R}^L_{1/2}$ is initially defined from $L^2(\mathbb{R}^{1+n}_+)$ to $L^2_{\mathsf{loc}}((0,\infty); L^2)$ by the L^2-valued Bochner integrals

$$\mathcal{R}^L_{1/2}(F)(t) := \int_0^t e^{-(t-s)L} \operatorname{div} F(s)\, ds, \quad t > 0.$$

6.3.1 Critical exponents

To precise our results, we first extend the four critical numbers $p_\pm(L)$ and $q_\pm(L)$ defined in Section 6.1.3 to the range of homogeneous Hardy–Sobolev spaces $\dot{H}^{s,p}$. For $-1 \leq s \leq 1$, define $p_\pm(s, L)$ as

$$\frac{1}{p_-(s, L)} := \begin{cases} \frac{1}{p_-(L)} + \frac{s}{n} & \text{if } 0 \leq s \leq 1 \\ \frac{1+s}{p_-(L)} - \frac{s}{q_+(L^*)'} & \text{if } -1 \leq s \leq 0 \end{cases}, \tag{6.32}$$

and

$$p_+(s, L) := \max\{p_-(-s, L^*), 1\}'. \tag{6.33}$$

Notice that $p_\pm(0, L) = p_\pm(L)$, $p_-(-1, L) = q_+(L^*)' \in [1, 2)$, and that $p_+(1, L) = q_+(L) \in (2, \infty]$. In particular, $p_-(s, -\Delta) = \frac{n}{n+s+1}$ and $p_+(s, -\Delta) = \infty$.

We also introduce several other numbers which will parametrize our results. For coherence, we still use the parameter β, whose relation to the regularity exponent s is given by

$$\boxed{s = 2\beta + 1}$$

with $\beta > -1$ and no upper restriction. Define

$$\beta(L) := -\frac{1}{2} - \frac{n}{2}\left(\frac{1}{p_-(L)} - 1\right) \geq -1. \tag{6.34}$$

Note that $\beta(L) \geq -1/2$ if and only if $p_-(L) \geq 1$. For $\beta > -1$, we introduce the numbers $p_L(\beta) \in (0,2)$ given by

$$p_L(\beta) := \frac{np_-(L)}{n + (2\beta+1)p_-(L)}, \tag{6.35}$$

and also $\widetilde{p}_L(\beta) \in (0,2)$ so that:

(i) When $p_-(L) \geq 1$, it is given by

$$\widetilde{p}_L(\beta) := \begin{cases} p_L(\beta) & \text{if } \beta \geq -1/2 \\ p_-(2\beta+1, L) & \text{if } -1 < \beta < -1/2 \end{cases}. \tag{6.36}$$

(ii) When $p_-(L) < 1$, it is given by

$$\widetilde{p}_L(\beta) := \begin{cases} p_L(\beta) & \text{if } \beta \geq \beta(L) \\ \frac{(\beta(L)+1)q_+(L^*)}{(\beta(L)+1)q_+(L^*)+\beta-\beta(L)} & \text{if } -1 < \beta < \beta(L) \end{cases}. \tag{6.37}$$

Remark that $\widetilde{p}_L(\beta(L)) = 1$ and $\widetilde{p}_L(-1) = q_+(L^*)' = p_-(-1, L)$. For the negative Laplacian, as $p_-(-\Delta) = \frac{n}{n+1}$, we have

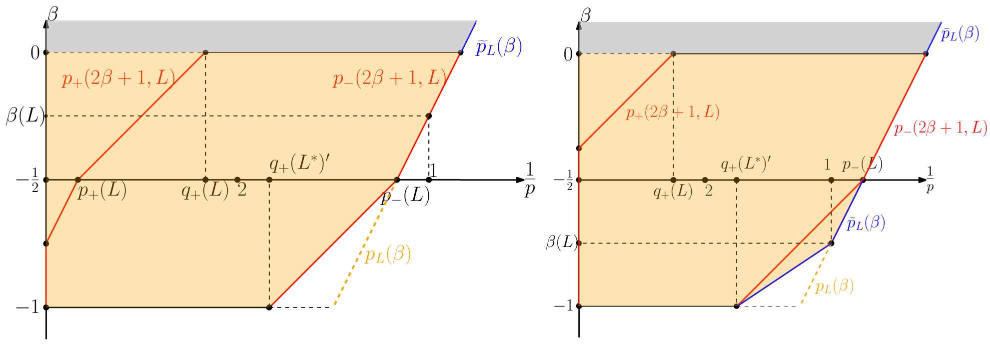
$$\widetilde{p}_{-\Delta}(\beta) = p_{-\Delta}(\beta) = \frac{n}{n+2\beta+2}, \quad \forall \beta > -1.$$

Observe that for any L,

$$\widetilde{p}_L(\beta) \geq p_L(\beta) \geq \frac{n}{n+2\beta+2}.$$

To illustrate these exponents, we give graphic representations in Figure 6.1, distinguishing the two cases $p_-(L) \geq 1$ and $p_-(L) < 1$. In these figures, we write p for $1/p$ to ease the presentation. When $p < 2$, we use red color for the graph of $p_-(2\beta+1, L)$, blue for that of $\widetilde{p}_L(\beta)$, and orange for that of $p_L(\beta)$. The orange shaded trapezoids are the regions of well-posedness for $\dot{H}^{2\beta+1,p}$-initial data, while the gray shaded is for constant initial data. Parallel lines to the orange one are lines of embedding for Hardy–Sobolev spaces and weighted tent spaces going downward.

Interestingly, the (smaller) set delimited by the red lines (excluded) and the black lines (included) has a special signification and is called the *identification range* for the operator L, and $\mathcal{E}_L$ defined in (6.5) can be extended to a semigroup on $\dot{H}^{s,p}$. The triangular region between the red and blue segments in the case $p_-(L) < 1$ is a region of embedding of an adapted space into $\dot{H}^{s,p}$ without knowing identification. See Section 6.4 and in particular Proposition 6.36. [2]



Case $p_-(L) \geq 1$ Case $p_-(L) < 1$

Figure 6.1: Regions of well-posedness

6.3.2 Main results

We use the the critical exponent $\widetilde{p}_L(\beta)$ defined in (6.36) and (6.37).

Theorem 6.22 (Extension of $\mathcal{R}^L_{1/2}$)**.** *Let $\beta > -1$ and $\widetilde{p}_L(\beta) < p \leq \infty$. Then $\mathcal{R}^L_{1/2}$ extends to a bounded operator from $T^p_{\beta+1/2}$ to $T^p_{\beta+1}$, also denoted by $\mathcal{R}^L_{1/2}$. Moreover, the following properties hold for any $F \in T^p_{\beta+1/2}$ and $u := \mathcal{R}^L_{1/2}(F)$.*

(a) (Regularity) *u lies in $T^p_{\beta+1}$ and ∇u lies in $T^p_{\beta+1/2}$ with*

$$\|u\|_{T^p_{\beta+1}} \lesssim \|F\|_{T^p_{\beta+1/2}}, \quad \|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|F\|_{T^p_{\beta+1/2}}.$$

(b) (Explicit formulae) *Define $\widetilde{F} := (A - \mathbb{I})\nabla\mathcal{R}^L_{1/2}(F) + F$. Then*

$$u = \mathcal{R}^{-\Delta}_{1/2}(\widetilde{F}) = \operatorname{div} \mathcal{L}^{-\Delta}_1(\widetilde{F}) \quad \text{in } \mathscr{D}'(\mathbb{R}^{1+n}_+),$$

where $\mathcal{L}^{-\Delta}_1$ is defined in (6.6).

[2] In the case where $p > 2$, the red colored broken lines for the values of $p_+(2\beta+1, L)$ defined by (6.33) are obtained by symmetry about the point $(1/p, \beta) = (1/2, -1/2)$ from the ones for $\max\{p_-(-2\beta-1, L^*), 1\}$. Thus it depends on the value of $p_-(L^*)$. In the first (resp. second) picture, this corresponds to having $p_-(L^*) \geq 1$ (resp. $p_-(L^*) < 1$). As the values of $p_-(L)$ and $p_-(L^*)$ are independent, we should have in fact 4 such figures.

(c) *u is a global weak solution to the equation* $\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F$.

(d) (Continuity and trace) $u \in C([0,\infty); \mathscr{S}')$ *with* $u(0) = 0$.

When $t \to 0$, the convergence also occurs in the following spaces shown in Table 6.1, with arbitrary parameters $\delta > 0$, $q \in [p,\infty]$, and $s \in [-1, 2\beta+1]$, and a fixed one $r := \frac{np}{n-(2\beta+2)p} > 1$.

Table 6.1: Spaces for convergence of $\mathcal{R}^L_{1/2}(F)(t)$ as $t \to 0$.

Conditions	$\widetilde{p}_L(\beta) < p \le 2$	$2 < p \le \infty$
$\beta \ge -1/2$	L^p	$E^{-1,q}_\delta$
$-1 < \beta < -1/2$	$\begin{cases} \dot{H}^{-1,r} & \text{if } p \le 1 \\ \dot{H}^{s,p} & \text{if } 1 < p \le 2 \end{cases}$	$E^{-1,q}_\delta$

Consequently, u is a global weak solution to the Lions' equation (L).

The proof of (a) is presented in Section 6.3.3; the proof of (b) and (c) is exhibited in Section 6.3.4; and the proof of (d) is deferred to Section 6.3.5. Let us first give some remarks.

Remark 6.23. Property (d) allows us to make sense of $u(t)$ for any $t \ge 0$ in $\mathscr{S}'$. Besides, when $\beta \ge -1/2$ and $\widetilde{p}_L(\beta) < p < 1$, we also have trace in L^p, but this topology is not compatible with that of $\mathscr{S}'$, so one cannot identify the limits.

Remark 6.24 (Whitney trace). When $\beta > -1/2$, there is another notion of trace (valid for any function $u \in T^p_{\beta+1}$ for $0 < p \le \infty$) in the sense of taking the limit of averages on Whitney cubes as $t \to 0$ for a.e. $x \in \mathbb{R}^n$. More precisely, when $\beta > -1/2$, for a.e. $x \in \mathbb{R}^n$, we have

$$\lim_{t\to 0+} \left(\fint_{W(t,x)} |u(s,y)|^2 \, ds dy \right)^{1/2} = 0.$$

See Lemma 6.18 for the proof.

Remark 6.25. We mention that the bounded extension of $\mathcal{R}^L_{1/2}$ on tent spaces T^p_0 was studied in [AMP19, Proposition 2.12]. It was claimed there that $\mathcal{R}^L_{1/2}$ extends to a bounded operator from T^p_0 to the Kenig–Pipher space X^p for $0 < p \le \infty$. However, the proof has a gap and we do not know if their full conclusion is correct. Yet, as $T^p_{1/2}$ embeds into X^p, our Theorem 6.22 ensures that their proposition is still valid in some range of p, which in fact covers the usage of this proposition (only when $L = -\Delta$ or $p = 2$) in that paper.

Remark 6.26. As we shall see in Sections 6.3.4 and 6.3.5, the proofs of (b), (c), and (d) work for any $\beta > -1$ and $\frac{n}{n+2\beta+2} < p \le \infty$, for which the operators $\mathcal{R}^L_\kappa$ are bounded from $T^p_{\beta+1/2}$ to $T^p_{\beta+1/2+\kappa}$ for $\kappa = 0, 1/2$ ($\mathcal{R}^L_0$ is defined right below).

6.3.3 Extension of the Lions' operator

We prove Theorem 6.22 (a) using the following two lemmas requiring different methods. Define the operator $\mathcal{R}^L_0 \colon L^1_{\mathsf{c}}((0,\infty); W^{2,2}) \to L^\infty_{\mathsf{loc}}((0,\infty); L^2)$ by the L^2-valued Bochner integrals

$$\mathcal{R}^L_0(F)(t) := \int_0^t \nabla e^{-(t-s)L} \operatorname{div} F(s)\, ds, \quad \forall t > 0. \tag{6.38}$$

We also use $p_L(\beta)$ defined in (6.35) and set

$$\widetilde{p}^\flat_L(\beta) := \frac{nq_+(L^*)'}{n + 2(\beta+1)q_+(L^*)'}.$$

Lemma 6.27. *Let $\beta > -1$ and $\widetilde{p}^\flat_L(\beta) < p \le \infty$. The operator $\mathcal{R}^L_{1/2}$ (resp. $\mathcal{R}^L_0$) extends to a bounded operator from $T^p_{\beta+1/2}$ to $T^p_{\beta+1}$ (resp. $T^p_{\beta+1/2}$), also denoted by $\mathcal{R}^L_{1/2}$ (resp. $\mathcal{R}^L_0$).*

Lemma 6.28. (i) *Suppose $p_-(L) \ge 1$. Then the properties of $\mathcal{R}^L_{1/2}$ and $\mathcal{R}^L_0$ in Lemma 6.27 are also valid for $\beta > -1/2$ and $p_L(\beta) < p \le \infty$.*

(ii) *Suppose $p_-(L) < 1$. Then the properties of $\mathcal{R}^L_{1/2}$ and $\mathcal{R}^L_0$ in Lemma 6.27 are also valid for $\beta > \beta(L)$ and $p_L(\beta) < p \le 1$.*

The proofs are provided in Sections 6.3.3.1 and 6.3.3.2, respectively. Let us first show that given these two lemmas, Theorem 6.22 (a) holds.

Proof of Theorem 6.22 (a) assuming Lemmas 6.27 and 6.28. Boundedness of $\mathcal{R}^L_{1/2}$ and $\mathcal{R}^L_0$ comes from interpolation of tent spaces, thanks to Lemmas 6.27 and 6.28. The number $\widetilde{p}_L(\beta)$ is exactly designed for this interpolation argument. Using Proposition 6.29 (iii), the equality $\nabla\mathcal{R}^L_{1/2}(F) = \mathcal{R}^L_0(F)$ in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$ extends to all $F \in T^p_{\beta+1/2}$ by density (or weak*-density if $p = \infty$), so the estimate of $\nabla\mathcal{R}^L_{1/2}(F)$ follows. $\square$

6.3.3.1 Proof of Lemma 6.27

The proof is based on the theory of singular integral operators on tent spaces developed in Chapter 4. Notations and terminologies will follow there.

Proof of Lemma 6.27. It is a direct consequence of Proposition 4.12 and Corollary 4.14. The assumptions there are verified in Proposition 6.29 just below. □

Proposition 6.29 (SIO properties of $\mathcal{R}^L_{1/2}$ and $\mathcal{R}^L_0$). *Let $\beta > -1$.*

(i) *The operator $\mathcal{R}^L_{1/2}$ given by (6.7) belongs to $\mathrm{SIO}^{1/2+}_{2,q,\infty}$ when $q_+(L^*)' < q < p_+(L)$. It is bounded from $L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+1}(\mathbb{R}^{1+n}_+)$.*

(ii) *The operator $\mathcal{R}^L_0$ given by (6.38) extends to an operator (also denoted by $\mathcal{R}^L_0$) in $\mathrm{SIO}^{0+}_{2,q,\infty}$ for $q_+(L^*)' < q < q_+(L)$ that is bounded on $L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$.*

(iii) *For any $F \in L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$,*

$$\nabla \mathcal{R}^L_{1/2}(F) = \mathcal{R}^L_0(F) \quad \text{in } \mathscr{D}'(\mathbb{R}^{1+n}_+).$$

Proof. For (i), the off-diagonal estimates of $(t^{1/2}\nabla e^{-tL^*})_{t>0}$ imply that the function $(t,s) \mapsto \mathbb{1}_{\{s>t\}}(t,s)\nabla e^{-(s-t)L^*}$ lies in $\mathrm{SK}^{1/2}_{2,q,\infty}$ for $\max\{q_-(L^*),1\} < q < q_+(L^*)$ by definition of this class. Using duality of singular kernels (see Corollary 4.5), we get that the kernel of $\mathcal{R}^L_{1/2}$,

$$K_{1/2}(t,s) := \mathbb{1}_{\{t>s\}}(t,s) e^{-(t-s)L} \operatorname{div}$$

belongs to $\mathrm{SK}^{1/2}_{2,q,\infty}$ for $q_+(L^*)' < q < \max\{q_-(L^*),1\}' = p_+(L)$. Applying Lemma 4.6 yields that $\mathcal{R}^L_{1/2}$ is bounded from $L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$ to $L^2_{\beta+1}(\mathbb{R}^{1+n}_+)$ as $\beta + 1/2 > -1/2$. This proves (i).

For (ii), we construct the extension as follows. Define the operator $\mathcal{L}_0$ from $L^2((0,\infty); D(L))$ to $L^\infty_{\mathsf{loc}}((0,\infty); L^2)$ by the L^2-valued Bochner integrals

$$\mathcal{L}_0(f)(t) := \int_0^t L e^{-(t-s)L} f(s)\, ds, \quad \forall t > 0.$$

De Simon's theorem states that $\mathcal{L}_0$ extends to a bounded operator $\tilde{\mathcal{L}}_0$ on $L^2(\mathbb{R}^{1+n}_+)$ [dS64]. Moreover, [AA11a, Theorem 1.3] shows that $\tilde{\mathcal{L}}_0$ is bounded on $L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$ for any $\beta > -1$, and [AMP19, Proposition 2.5] reveals

$$\mathcal{R}^L_0(F) = \nabla L^{-1/2} \tilde{\mathcal{L}}_0 L^{-1/2} \operatorname{div} F$$

for any $F \in L^1((0,\infty); W^{2,2})$. Recall that $\nabla L^{-1/2}$ and $L^{-1/2}\operatorname{div}$ are bounded on $L^2(\mathbb{R}^n)$ (see [AHL+02]), so we define the bounded extension of $\mathcal{R}^L_0$ on $L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$ by

$$\mathcal{R}^L_0 := \nabla L^{-1/2} \tilde{\mathcal{L}}_0 L^{-1/2} \operatorname{div}.$$

Next, let us verify that its kernel

$$K_0(t,s) := \mathbb{1}_{\{t>s\}}(t,s) \nabla e^{-(t-s)L^*} \operatorname{div}$$

belongs to $\mathrm{SK}^0_{2,q,\infty}$ for $q_+(L^*)' < q < q_+(L)$. Indeed, it can be written as

$$K_0(t,s) = \mathbb{1}_{\{t>s\}}(t,s)\nabla e^{-\frac{1}{2}(t-s)L}\mathbb{1}_{\{t>s\}}(t,s)e^{-\frac{1}{2}(t-s)L}\,\mathrm{div},$$

where $(t,s) \mapsto \mathbb{1}_{\{t>s\}}(t,s)e^{-\frac{1}{2}(t-s)L}\,\mathrm{div}$ lies in $\mathrm{SK}^{1/2}_{2,q,\infty}$ for $q_+(L^*)' < q < p_+(L)$, and $(t,s) \mapsto \mathbb{1}_{\{t>s\}}(t,s)\nabla e^{-\frac{1}{2}(t-s)L}$ lies in $\mathrm{SK}^{1/2}_{2,q,\infty}$ for $\max\{q_-(L),1\} < q < q_+(L)$. Hence, we obtain $K_0 \in \mathrm{SK}^0_{2,q,\infty}$ for $q_+(L^*)' < q < q_+(L)$ by composition.

Finally, for the representation, we have to show that for any $F \in L^2_{\mathsf{b}}(\mathbb{R}^{1+n}_+)$ and a.e. $(t,x) \in (\mathbb{R}^{1+n}_+ \setminus \pi(F))$,

$$\mathcal{R}^L_0(F)(t,x) = \int_0^t (\nabla e^{-(t-s)L}\,\mathrm{div}\,F(s))(x)\,ds. \tag{6.39}$$

The proof of (6.39), as well as the identity in (iii) follows by a verbatim adaptation of Lemma 6.16. Details are left to the reader. □

6.3.3.2 Proof of Lemma 6.28

The proof consists of two parts. For $p > 1$, we argue by duality. For $p \le 1$, in the spirit of [AP25, Theorem 3.2], we use atomic decomposition in weighted tent spaces. Recall that for $0 < p \le 1$, a measurable function a on $\mathbb{R}^{1+n}_+$ is called a T^p_β-*atom*, if there exists a ball $B \subset \mathbb{R}^n$ so that $\mathrm{supp}(a) \subset [0, r(B)^2] \times B$, and

$$\|a\|_{L^2_\beta(\mathbb{R}^{1+n}_+)} \le |B|^{-[p,2]}.$$

Such a ball B is said to be *associated* to a. Write $r := r(B)$, $C_0 := 2B$, and $C_j := 2^{j+1}B \setminus 2^jB$ for $j \ge 1$. For $j \ge 4$, define

$$\begin{aligned}
M^{(1)}_j &:= \big(0, (2^3r)^2\big] \times C_j,\\
M^{(2)}_j &:= \big((2^3r)^2, (2^jr)^2\big) \times C_j,\\
M^{(3)}_j &:= \big[(2^jr)^2, (2^{j+1}r)^2\big) \times 2^{j+1}B.
\end{aligned}$$

The next lemma shows molecular decay for $\mathcal{R}^L_{1/2}$ acting on atoms.

Lemma 6.30 (Molecular decay when $p_-(L) \ge 1$)**.** *Assume $p_-(L) \ge 1$. Let $\beta > -1/2$, $0 < p \le 1$, and $p_-(L) < q < 2$. There exists a constant $c > 0$ depending on L and q, so that for any $T^p_{\beta+1/2}$-atom a with an associated ball $B \subset \mathbb{R}^n$, the following estimates hold for $u := \mathcal{R}^L_{1/2}(a)$ and any $j \ge 4$,*

$$\|u\|_{L^2_{\beta+1}(M^{(1)}_j)} \lesssim 2^{jn[p,2]}e^{-c2^{2j}}|2^{j+1}B|^{[2,p]}, \tag{6.40}$$

$$\|u\|_{L^2_{\beta+1}(M^{(2)}_j)} \lesssim 2^{-j(2\beta+1+n[q,p])}|2^{j+1}B|^{[2,p]}, \tag{6.41}$$

$$\|u\|_{L^2_{\beta+1}(M^{(3)}_j)} \lesssim 2^{-j(2\beta+1+n[q,p])}|2^{j+1}B|^{[2,p]}. \tag{6.42}$$

The implicit constants are independent of j and B.

Proof. Note that since $a \in L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$, Proposition 6.29 (i) ensures that $u := \mathcal{R}^L_{1/2}(a)$ lies in $L^2_{\beta+1}(\mathbb{R}^{1+n}_+)$, and for a.e. $(t,x) \in \mathbb{R}^{1+n}_+$,

$$u(t,x) = \int_0^t (e^{-(t-s)L} \operatorname{div} a(s))(x)\, ds.$$

This allows us to use duality to get estimates of $u(t)$ for a.e. $t > 0$. To this end, we fix $\phi \in C^\infty_{\mathsf{c}}(\mathbb{R}^n)$ and set $v_t(s,y) := (e^{-(t-s)L^*}\phi)(y)$. We get

$$\langle u(t), \phi \rangle = \int_0^t \int_{\mathbb{R}^n} a(s,y) \cdot \overline{\nabla v_t(s,y)}\, dsdy.$$

Using the properties of a and Cauchy–Schwarz inequality, we have

$$|\langle u(t), \phi\rangle| \le |B|^{[2,p]} \left(\int_0^{\min\{r^2,t\}} \int_B s^{2\beta+1} |\nabla v_t(s,y)|^2 \, dsdy \right)^{1/2}. \tag{6.43}$$

The first inequality (6.40) follows from the estimate

$$\|u(t)\|_{L^2(C_j)} \lesssim |B|^{[2,p]} t^{\beta+1/2} e^{-\frac{c(2^j r)^2}{t}}, \quad 0 < t \le (2^3 r)^2, \tag{6.44}$$

by integrating the square of both sides over $0 < t \le (2^3r)^2$. To prove (6.44), one applies (6.43) for $\phi \in C^\infty_{\mathsf{c}}(\mathbb{R}^n)$ and $\operatorname{supp}(\phi) \subset C_j$. The $L^2 - L^2$ off-diagonal estimates of $(t^{1/2}\nabla e^{-tL^*})$ yield that there exists $c_1 > 0$ only depending on L so that for any $c \in (0, c_1)$,

$$\int_0^t \int_B s^{2\beta+1} |\nabla v_t(s,y)|^2 \, dsdy \lesssim t^{2\beta+1} e^{-\frac{2c(2^j r)^2}{t}} \|\phi\|_2^2.$$

Thus, one obtains (6.44).

The second inequality (6.41) is obtained similarly from

$$\|u(t)\|_{L^2(C_j)} \lesssim |B|^{[q,p]} r^{2\beta+1} t^{-\frac{n}{2}[2,q']} e^{-\frac{c(2^j r)^2}{t}}, \quad (2^3 r)^2 < t < (2^j r)^2.$$

To this end, we use again (6.43) for $\phi \in C^\infty_{\mathsf{c}}(\mathbb{R}^n)$ and $\operatorname{supp}(\phi) \subset C_j$. Note that v_t is a weak solution to the backward equation $-\partial_s v - \operatorname{div}(A^* \nabla v) = 0$ on $(-\infty, t) \times \mathbb{R}^n$. As $\beta > -1/2$, we infer from Caccioppoli's inequality (cf. Lemma 6.3) that

$$\begin{aligned}\int_0^{r^2} \int_B s^{2\beta+1} |\nabla e^{-(t-s)L^*}\phi(y)|^2 \, dsdy &\le (r^2)^{2\beta+1} \int_0^{r^2} \int_B |\nabla e^{-(t-s)L^*}\phi(y)|^2 \, dsdy \\ &\lesssim (r^2)^{2\beta} \int_0^{2r^2} \int_{2B} |e^{-(t-s)L^*}\phi(y)|^2 \, dsdy.\end{aligned}$$

Using Hölder's inequality and $L^2 - L^{q'}$ off-diagonal estimates of (e^{-tL^*}), we get $c_2 > 0$, depending on L and q, so that for any $c \in (0, c_2)$,

$$\begin{aligned}\|\mathbb{1}_{2B} e^{-(t-s)L^*}\phi\|_2 &\lesssim |B|^{[2,q']} \|\mathbb{1}_{2B} e^{-(t-s)L^*}\phi\|_{q'} \\ &\lesssim |B|^{[2,q']} t^{-\frac{n}{2}[2,q']} e^{-\frac{c(2^j r)^2}{t}} \|\phi\|_2.\end{aligned}$$

Hence, if $\|\phi\|_2 = 1$, then plugging the above estimates in (6.43) implies

$$\begin{aligned}
|\langle u(t), \phi\rangle|^2 &\leq |B|^{2[2,p]} \int_0^{r^2} \int_B s^{2\beta+1} |\nabla v_t(s,y)|^2 \, dsdy \\
&\lesssim |B|^{2[2,p]} r^{4\beta} \int_0^{2r^2} \|\mathbb{1}_{2B} e^{-(t-s)L^*} \phi\|_2^2 \, ds \\
&\lesssim |B|^{2[q,p]} r^{2(2\beta+1)} t^{-n[2,q']} e^{-\frac{2c(2^j r)^2}{t}}
\end{aligned}$$

as desired.

The third inequality (6.42) follows from a similar argument as that of (6.41) with the same constant c_2. Details are left to the reader. The assertions hence hold for $0 < c < \min\{c_1, c_2\}$, which only depends on L and q. $\square$

When $p_-(L) < 1$, we have even better decay for all $\beta > -1$, using pointwise estimates and Hölder continuity of weak solutions to the backward equation.

Corollary 6.31 (Molecular decay when $p_-(L) < 1$)**.** *Suppose $p_-(L) < 1$. Let $\beta > -1$, $0 < p \leq 1$, and $0 < \eta < n(\frac{1}{p_-(L)} - 1)$. There exists a constant $c > 0$ only depending on L, so that for any $T^p_{\beta+1/2}$-atom a with an associated ball $B \subset \mathbb{R}^n$, the following estimates hold for $u := \mathcal{R}^L_{1/2}(a)$ and any $j \geq 4$,*

$$\begin{aligned}
\|u\|_{L^2_{\beta+1}(M_j^{(1)})} &\lesssim 2^{jn[p,2]} e^{-c2^{2j}} |2^{j+1}B|^{[2,p]}, \\
\|u\|_{L^2_{\beta+1}(M_j^{(2)})} &\lesssim 2^{-j(2\beta+1+\eta-n[p,1])} |2^{j+1}B|^{[2,p]}, \\
\|u\|_{L^2_{\beta+1}(M_j^{(3)})} &\lesssim 2^{-j(2\beta+1+\eta-n[p,1])} |2^{j+1}B|^{[2,p]}.
\end{aligned}$$

The implicit constants are independent of j and B.

Proof. Fix $c > 0$ as the constant given by the $L^2 - L^2$ off-diagonal estimates of $(t^{1/2} \nabla e^{-tL^*})$, which only depends on L. The first inequality is the same as in Lemma 6.30. We only need to show the second, and the third follows analogously. The second inequality is also obtained by integrating the square of

$$\|u(t)\|_{L^2(C_j)} \lesssim |B|^{[2,p]} r^{\frac{n}{2}+2\beta+1+\eta} t^{-(\frac{n}{4}+\frac{\eta}{2})} e^{-\frac{c(2^j r)^2}{t}}, \quad (2^3 r)^2 < t < (2^j r)^2. \tag{6.45}$$

To prove it, we use (6.43) again for $\phi \in C^\infty_c(\mathbb{R}^n)$ with $\|\phi\|_2 = 1$ and $\operatorname{supp}(\phi) \subset C_j$. When $p_-(L) < 1$, we know that (e^{-tL^*}) satisfies $L^2 - L^\infty$ off-diagonal estimates, and it is uniformly bounded from L^2 to $\dot{\Lambda}^\eta$ for $0 < \eta < n(\frac{1}{p_-(L)} - 1)$, where $\dot{\Lambda}^\eta$ is the *homogeneous η-Hölder space*, see [AE23a, Chapter 14]. In particular, $v_t(s,y) := (e^{-(t-s)L^*}\phi)(y)$ has pointwise values. Set

$$\widetilde{v}_t(s,y) := v_t(s,y) - v_t(0,0) = (v_t(s,y) - v_t(s,0)) + (v_t(s,0) - v_t(0,0)).$$

and we estimate each term separately. For the first term, we use Hölder continuity. For the second, we write it as $\int_0^s (L^* e^{-(t-\tau)L^*}\phi)(0)\,d\tau$ and use $L^2 - L^\infty$ boundedness of $(tL^* e^{-tL^*})$. In summary, as $\mathrm{dist}(\mathrm{supp}(\phi), B) \sim 2^j r$, we obtain that for any $0 < \eta < n(\frac{1}{p_-(L)} - 1)$,

$$\sup_{0<s<2r^2, y\in 2B} |\widetilde{v}_t(s,y)| \lesssim t^{-(\frac{n}{4}+\frac{\eta}{2})} e^{-\frac{c(2^j r)^2}{t}} r^\eta \|\phi\|_2. \tag{6.46}$$

By analyticity of the semigroup again, one finds that $\partial_s \widetilde{v}_t(s,y)$ satisfies the similar estimate with an extra factor $(t-s)^{-1} \sim t^{-1}$.

Let χ be in $C^\infty([0,\infty))$ with $\mathrm{supp}(\chi) \subset [0, \frac{3}{2}r^2]$, $0 \le \chi \le 1$, $\chi = 1$ on $[0, r^2]$, and $\|\chi'\|_\infty \lesssim r^{-2}$. Using integration by parts in variable s, we get

$$\begin{aligned}
\int_0^{r^2} \int_B |\nabla \widetilde{v}_t(s,y)|^2 s^{2\beta+1}\, dsdy &\le \int_0^{\frac{3}{2}r^2} \int_B \chi(s) |\nabla \widetilde{v}_t(s,y)|^2 s^{2\beta+1}\, dsdy \\
&\lesssim \int_0^{\frac{3}{2}r^2} \int_B |\chi'(s)| |\nabla \widetilde{v}_t(s,y)|^2 s^{2\beta+2}\, dsdy \\
&\quad + \int_0^{\frac{3}{2}r^2} \int_B \chi(s) |\nabla \widetilde{v}_t(s,y)| |\nabla \partial_s \widetilde{v}_t(s,y)| s^{2\beta+2}\, dsdy =: I_1 + I_2.
\end{aligned}$$

For I_1, note that $\widetilde{v}_t$ is also a weak solution to the backward equation $-\partial_s v - \mathrm{div}(A^* \nabla v) = 0$ on $(-\infty, t) \times \mathbb{R}^n$. As $|\chi'| \lesssim r^{-2}$ and $2\beta + 2 > 0$, we get from Caccioppoli's inequality and (6.46) that

$$\begin{aligned}
I_1 &\lesssim r^{4\beta+2} \int_0^{\frac{3}{2}r^2} \int_B |\nabla \widetilde{v}_t|^2 \lesssim r^{4\beta} \int_0^{2r^2} \int_{2B} |\widetilde{v}_t|^2 \\
&\lesssim |B| r^{4\beta+2+2\eta} t^{-(\frac{n}{2}+\eta)} e^{-\frac{2c(2^j r)^2}{t}} \|\phi\|_2^2.
\end{aligned}$$

For I_2, using again Caccioppoli's inequality and the above estimate for $\partial_s \widetilde{v}_t(s,y)$, we find

$$\begin{aligned}
I_2 &\lesssim r^{4\beta+4} \left(\int_0^{\frac{3}{2}r^2} \int_B |\nabla \widetilde{v}_t|^2 \right)^{1/2} \left(\int_0^{\frac{3}{2}r^2} \int_B |\nabla \partial_s \widetilde{v}_t|^2 \right)^{1/2} \\
&\lesssim |B| r^{4\beta+4+2\eta} t^{-(\frac{n}{2}+1+\eta)} e^{-\frac{2c(2^j r)^2}{t}} \|\phi\|_2^2.
\end{aligned}$$

Gathering these estimates, we easily obtain (6.45). □

Proof of Lemma 6.28. We begin with (i) and split the discussion into two cases.

Case 1: $\max\{p_L(\beta), 1\} < p \le \infty$ First consider $\mathcal{R}^L_{1/2}$. By density (or weak*-density when $p = \infty$), it suffices to show that for any $F \in T^p_{\beta+1/2} \cap L^2(\mathbb{R}^{1+n}_+)$,

$$\|\mathcal{R}^L_{1/2}(F)\|_{T^p_{\beta+1}} \lesssim \|F\|_{T^p_{\beta+1/2}}. \tag{6.47}$$

To prove (6.47), we use duality. Fix $\phi \in C_c^\infty(\mathbb{R}^{1+n}_+)$. By Fubini's theorem, we get

$$\langle \mathcal{R}^L_{1/2}(F), \phi \rangle_{L^2(\mathbb{R}^{1+n}_+)} = \int_0^\infty \int_{\mathbb{R}^n} F(s,y) \left(\int_s^\infty \overline{\nabla e^{-(t-s)L^*} \phi(t)}(y) \, dt \right) dsdy.$$

Define

$$v(s,y) := (\mathcal{L}^L_1)^*(\phi)(s,y) = \int_s^\infty e^{-(t-s)L^*} \phi(t) \, dt.$$

Here the adjoint is taken with respect to the $L^2(\mathbb{R}^{1+n}_+)$-duality. As $1 \leq p' < \infty$, by duality of tent spaces, (6.47) follows from the estimate

$$\|\nabla v\|_{T^{p'}_{-(\beta+1/2)}} \lesssim \|\phi\|_{T^{p'}_{-(\beta+1)}}. \tag{6.48}$$

Indeed, observe that $v \in L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$ is a weak solution to the backward equation $-\partial_s v - \operatorname{div}(A^* \nabla v) = \phi$ on $\mathbb{R}^{1+n}$. Applying Corollary 6.4 leads to

$$\|\nabla v\|_{T^{p'}_{-(\beta+1/2)}} \lesssim \|v\|_{T^{p'}_{-\beta}} + \|\phi\|_{T^{p'}_{-\beta-1}}.$$

As we assume $p_-(L) \geq 1$, we obtain that for $\beta > -1/2$ and $1 \leq p' < (\max\{1, p_L(\beta)\})'$,

$$\|v\|_{T^{p'}_{-\beta}} \lesssim \|\phi\|_{T^{p'}_{-\beta-1}}.$$

Namely, we also apply the theory of singular integral operators on tent spaces. First, Lemma 6.13 says $\mathcal{L}^L_1$ belongs to $\mathrm{SIO}^{1+}_{2,q,\infty}$ for $\max\{p_-(L), 1\} < q < p_+(L)$. Then by duality, Proposition 4.11 implies that $(\mathcal{L}^L_1)^*$ belongs to $\mathrm{SIO}^{1-}_{2,q,\infty}$ for $\max\{p_-(L^*), 1\} < q < p_+(L^*) = p_-(L)'$. Thus, applying Proposition 4.13 and Corollary 4.15 proves the estimate (6.48) as

$$\|v\|_{T^{p'}_{-\beta}} = \|(\mathcal{L}^L_1)^*(\phi)\|_{T^{p'}_{-\beta}} \lesssim \|\phi\|_{T^{p'}_{-\beta-1}}.$$

This finally proves (6.47) and hence shows that $\mathcal{R}^L_{1/2}$ extends to a bounded operator from $T^p_{\beta+1/2}$ to $T^p_{\beta+1}$.

Next, consider $\mathcal{R}^L_0$. Fix $F \in T^p_{\beta+1/2} \cap L^2(\mathbb{R}^{1+n}_+)$. Proposition 6.29 (iii) says $\mathcal{R}^L_0(F) = \nabla \mathcal{R}^L_{1/2}(F)$ in $L^2(\mathbb{R}^{1+n}_+)$. Meanwhile, by Proposition 6.5, $u = \mathcal{R}^L_{1/2}(F)$ is a global weak solution to the equation $\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F$. Applying Corollary 6.4 and (6.47) yields that for any $F \in T^p_{\beta+1/2} \cap L^2(\mathbb{R}^{1+n}_+)$,

$$\|\mathcal{R}^L_0(F)\|_{T^p_{\beta+1/2}} = \|\nabla \mathcal{R}^L_{1/2}(F)\|_{T^p_{\beta+1/2}} \lesssim \|\mathcal{R}^L_{1/2}(F)\|_{T^p_{\beta+1}} + \|F\|_{T^p_{\beta+1/2}} \lesssim \|F\|_{T^p_{\beta+1/2}}.$$

Then the bounded extension of $\mathcal{R}^L_0$ follows by density.

Case 2: $p_L(\beta) < p \leq 1$ Note that by definition of $\beta(L)$ (cf. (6.34)), this case only occurs when $\beta > \beta(L)$. We first consider the extension of $\mathcal{R}^L_{1/2}$. Thanks to Corollary 3.6, it is enough to show $\mathcal{R}^L_{1/2}$ is uniformly bounded on $T^p_{\beta+1/2}$-atoms.

Let us verify this. Let a be a $T^p_{\beta+1/2}$-atom, $B \subset \mathbb{R}^n$ be a ball associated to a, and $u := \mathcal{R}^L_{1/2}(a)$. Denote by $Q_0 := \big(0, (2^3 r(B))^2\big) \times 2^3 B$. As $\beta + 1/2 > -1/2$, Proposition 6.29 (i) implies

$$\|u\|_{L^2_{\beta+1}(Q_0)} \lesssim \|a\|_{L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)} \lesssim |2^3 B|^{[2,p]}. \tag{6.49}$$

Pick $q < 2$ sufficiently close to $p_-(L)$ so that

$$\frac{np_-(L)}{n+(2\beta+1)p_-(L)} < \frac{nq}{n+(2\beta+1)q} < p \leq 1. \tag{6.50}$$

As $\beta > \beta(L) \geq -1/2$, we may apply Lemma 6.30 as well as the estimates given by (6.49) to get

$$\|u\|^p_{T^p_{\beta+1}} \lesssim 1 + \sum_{j\geq 4} 2^{-jp(2\beta+1+n[q,p])} \leq C$$

for some constant $C < \infty$ by (6.50). The assertion hence holds in this case.

Next, we consider the bounded extension of $\mathcal{R}^L_0$. In fact, one can apply Caccioppoli's inequality as in Corollary 6.4 to establish the same molecular decay estimates as in Lemma 6.30 for ∇u in $L^2_{\beta+1/2}(M^{(i)}_j)$ $(1 \leq i \leq 3)$. Meanwhile, using Proposition 6.29 (ii), one can also get as an alternative of (6.49)

$$\|\nabla u\|_{L^2_{\beta+1/2}(Q_0)} \lesssim \|a\|_{L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)} \lesssim |2^3 B|^{[2,p]}.$$

It is then enough to repeat the above computation. Details are left to the reader. This proves (i).

To prove (ii), as $p \leq 1$, we can use the argument in Case 2, using (6.49) directly and replacing the estimates of Lemma 6.30 by those in Corollary 6.31. Picking $\eta > 0$ sufficiently close to $n(\frac{1}{p_-(L)} - 1)$ so that

$$\frac{np_-(L)}{n+(2\beta+1)p_-(L)} < \frac{n}{n+2\beta+1+\eta} < p \leq 1,$$

we obtain some constant $C < \infty$ so that

$$\|u\|^p_{T^p_{\beta+1}} \lesssim 1 + \sum_{j\geq 4} 2^{-jp(2\beta+1+\eta-n[p,1])} \leq C.$$

This proves bounded extension of $\mathcal{R}^L_{1/2}$.

Bounded extension of $\mathcal{R}^L_0$ follows by the same argument as in the above Case 2, using Corollary 6.31. This completes the proof. $\square$

6.3.4 Existence

Let us prove Theorem 6.22 (b) and (c).

Proof of Theorem 6.22 (b) and (c). First, Corollary 6.7 (i) says that (b) holds for $F \in T^p_{\beta+1/2} \cap L^2(\mathbb{R}^{1+n}_+)$. Also note that all the operators involved have bounded extensions to $T^p_{\beta+1/2}$. Indeed, for $\mathcal{R}^L_{1/2}$, $\nabla\mathcal{R}^L_{1/2}$, and $\mathcal{R}^{-\Delta}_{1/2}$, it has been shown in (a); For $\mathcal{L}^{-\Delta}_1$, Theorem 6.11 says that it is bounded from $T^p_{\beta+1/2}$ to $T^p_{\beta+3/2}$. We thus conclude by density (or weak*-density if $p=\infty$), since all the weighted tent spaces embed into $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$, hence into $\mathscr{D}'(\mathbb{R}^{1+n}_+)$.

For (c), we infer from (a) that for any $F \in T^p_{\beta+1/2}$, both $u = \mathcal{R}^L_{1/2}(F)$ and ∇u lie in $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$, so $u \in L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$. Recall that Proposition 6.5 says for any $F \in T^p_{\beta+1/2} \cap L^2(\mathbb{R}^{1+n}_+)$, u is a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F$. The same density argument concludes the verification for any $F \in T^p_{\beta+1/2}$. $\square$

6.3.5 Continuity and traces

In this section, we exhibit the proof of Theorem 6.22 (d). We first prepare a lemma allowing us to reduce the proof to the case $p > 1$.

Lemma 6.32. *Let $\beta > -1$ and $\widetilde{p}_L(\beta) < p \le 1$. There exist $\beta_0 > -1$ and $\max\{\widetilde{p}_L(\beta_0), 1\} < p_0 < 2$ so that $T^p_{\beta+1/2}$ embeds into $T^{p_0}_{\beta_0+1/2}$.*

Proof. The crucial point is the embedding theorem for weighted tent spaces, recalled in Proposition 3.10. In parabolic scaling, it asserts that $T^p_{\beta+1/2}$ embeds into $T^{p_0}_{\beta_0+1/2}$ if

$$\beta_0 < \beta, \quad p < p_0, \quad 2\beta_0 + 1 - \frac{n}{p_0} = 2\beta + 1 - \frac{n}{p}.$$

Let $\gamma < \beta$ be the number so that

$$2\gamma + 1 - n = 2\beta + 1 - \frac{n}{p}.$$

We claim that $\gamma > -1$ and $\widetilde{p}_L(\gamma) < 1$. If it holds, then by perturbation, we can take $\beta_0 > -1$ sufficiently close to γ and the corresponding $p_0 > 1$. The claim is elementary using the definitions of the exponents in the introduction but we provide an argument for the sake of completeness.

By definition (see also Figure 6.1), we already observe that:

(i) $\beta(L) \ge -1/2 \iff p_-(L) \ge 1$;

(ii) $p_L(\beta) \le \widetilde{p}_L(\beta)$, and the equality holds if and only if $\beta \ge \min\{\beta(L), -1/2\}$.

This implies

$$\beta > -1 \text{ and } \widetilde{p}_L(\beta) < 1 \Longleftrightarrow \beta > \beta(L).$$

Indeed, if the right-hand side holds, then $\beta > \beta(L) \geq -1$ and $\widetilde{p}_L(\beta) = p_L(\beta) < p_L(\beta(L)) = 1$. Conversely, assume the left-hand side. When $p_-(L) < 1$, $\widetilde{p}_L(\beta) < 1$ implies $\beta > \beta(L)$. When $p_-(L) \geq 1$, by construction, $\widetilde{p}_L(\beta) < 1$ implies $\beta > -1/2 = \min\{\beta(L), -1/2\}$, so $\widetilde{p}_L(\beta) = p_L(\beta) < 1$. We thus infer $\beta > \beta(L)$.

It remains to verify $\gamma > \beta(L)$. Remark that for any $\delta > -1$ and $r > 0$,

$$r > \frac{np_-(L)}{n + (2\delta + 1)p_-(L)} \Longleftrightarrow 2\delta + 1 - \frac{n}{r} > -\frac{n}{p_-(L)}.$$

As $\frac{np_-(L)}{n+(2\beta+1)p_-(L)} \leq \widetilde{p}_L(\beta) < p \leq 1$, we have $2\gamma+1-n = 2\beta+1-\frac{n}{p} > -\frac{n}{p_-(L)}$. By definition of $\beta(L)$, this exactly says $\gamma > \beta(L)$ as desired. □

Proof of Theorem 6.22 (d). We first show that $u \in C((0,\infty); \mathscr{S}')$. Since u lies in $L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$, using the equation (by (c)), we get $\partial_t u \in L^2_{\mathsf{loc}}((0,\infty); W^{-1,2}_{\mathsf{loc}})$. Next, fix $t > 0$, $\phi \in \mathscr{S}$, and pick a, b so that $0 < a < t < b$. One can infer from Lemma 5.23 that there exists a sufficiently large $N > 0$ so that

$$\|\mathbb{1}_{(a,b)} \nabla \phi\|_{(T^p_{\beta+1/2})'} \lesssim_{a,b} \mathcal{P}_N(\phi), \tag{6.51}$$

where $\mathcal{P}_N$ is the norm on $\mathscr{S}$ given by (5.8). Then for $s \in (a,t)$, we apply the equation to get

$$\begin{aligned}
|\langle u(t) - u(s), \phi \rangle| &\leq \left| \int_s^t \langle \partial_\tau u(\tau), \phi \rangle d\tau \right| \\
&\leq \int_s^t \int_{\mathbb{R}^n} |A \nabla u||\nabla \phi| + \int_s^t \int_{\mathbb{R}^n} |F||\nabla \phi| \\
&\lesssim (\|\mathbb{1}_{(s,t)} \nabla u\|_{T^p_{\beta+1/2}} + \|\mathbb{1}_{(s,t)} F\|_{T^p_{\beta+1/2}}) \mathcal{P}_N(\phi).
\end{aligned}$$

For $p < \infty$, this proves the continuity as desired, since both $\|\mathbb{1}_{(s,t)} \nabla u\|_{T^p_{\beta+1/2}}$ and $\|\mathbb{1}_{(s,t)} F\|_{T^p_{\beta+1/2}}$ tend to 0 as $s \to t$. For $p = \infty$, we need finer estimates. Note that in this case, (6.51) can be refined as

$$\|\mathbb{1}_{(s,t)} \nabla \phi\|_{T^1_{-(\beta+1/2)}} \lesssim \left(\int_s^t \tau^{2\beta+1} d\tau \right)^{1/2} \mathcal{P}_N(\phi).$$

Repeating the above computation yields

$$|\langle u(t) - u(s), \phi \rangle| \lesssim \left(\int_s^t \tau^{2\beta+1} d\tau \right)^{1/2} (\|\nabla u\|_{T^\infty_{\beta+1/2}} + \|F\|_{T^\infty_{\beta+1/2}}) \mathcal{P}_N(\phi),$$

which also converges to 0 as $s \to t$.

Next, we consider the trace. We split the discussion in 4 cases.

Case 1: $\beta > -1$ and $2 < p \le \infty$ We use the formula in (b). Write $\widetilde{F} := (A - \mathbb{I})\nabla u + F$ so that $u = \operatorname{div} \mathcal{L}_1^{-\Delta}(\widetilde{F})$. Since $\widetilde{F} \in T^p_{\beta+1/2}$, we invoke Theorem 6.11 (e) to get that $\mathcal{L}_1^{-\Delta}(\widetilde{F})(t)$ tends to 0 in E^q_δ as $t \to 0$, so $u(t) = \operatorname{div} \mathcal{L}_1^{-\Delta}(\widetilde{F})(t)$ tends to 0 in $E^{-1,q}_\delta$ as $t \to 0$.

Case 2: $-1 < \beta < -1/2$ and $\max\{\widetilde{p}_L(\beta), 1\} < p \le 2$ By interpolation of Hardy–Sobolev spaces, it is enough to show that $u(t)$ tends to 0 as $t \to 0$ in $\dot{H}^{2\beta+1,p}$ and $\dot{H}^{-1,p}$.

For $\dot{H}^{-1,p}$, we also apply the formula in (b). Again, as $\widetilde{F} \in T^p_{\beta+1/2}$, Theorem 6.11 (e) says that $\mathcal{L}_1^{-\Delta}(\widetilde{F})(t)$ tends to 0 in L^p when $t \to 0$, so $u(t)$ tends to 0 in $\dot{W}^{-1,p} = \dot{H}^{-1,p}$ as $t \to 0$.

For $\dot{H}^{2\beta+1,p}$, we first claim that for any $t > 0$,

$$\|u(t)\|_{\dot{H}^{2\beta+1,p}} \lesssim \|\mathbb{1}_{(0,t)}\widetilde{F}\|_{T^p_{\beta+1/2}}. \tag{6.52}$$

If it holds, then it is clear that $u(t)$ tends to 0 in $\dot{H}^{2\beta+1,p}$ as $t \to 0$, since $\|\mathbb{1}_{(0,t)}\widetilde{F}\|_{T^p_{\beta+1/2}}$ tends to 0 as $t \to 0$.

Let us prove the claim (6.52) by duality. Fix $\varphi \in \mathscr{S}_\infty$. Define

$$g(s,x) := \mathbb{1}_{\{s<t\}}(s)\nabla e^{-(t-s)\Delta}\varphi(x).$$

Observe that

$$\|g\|_{T^{p'}_{-(\beta+1/2)}} \lesssim \|\varphi\|_{\dot{H}^{-(2\beta+1),p'}}. \tag{6.53}$$

Indeed, for $\widetilde{p}_L(\beta) < p < 2$, we have

$$\|g\|_{T^{p'}_{-(\beta+1/2)}} \le \left(\int_{\mathbb{R}^n} \left(\int_0^t \mathcal{M}(|\nabla e^{(t-s)\Delta}\varphi|^2)(x) s^{2\beta+1} ds \right)^{p'/2} dx \right)^{1/p'},$$

where $\mathcal{M}$ is the Hardy–Littlewood maximal operator on $\mathbb{R}^n$. By Minkowski's inequality and boundedness of $\mathcal{M}$ on $L^{p'/2}$, we further have

$$\begin{aligned}\|g\|_{T^{p'}_{-(\beta+1/2)}} &\lesssim \left(\int_0^t \|\nabla e^{(t-s)\Delta}\varphi\|^2_{p'} \, s^{2\beta+1} ds \right)^{1/2} \\ &\lesssim \left(\int_0^t (t-s)^{-2(\beta+1)} \|\varphi\|^2_{\dot{H}^{-(2\beta+1),p'}} s^{2\beta+1} ds \right)^{1/2} \lesssim \|\varphi\|_{\dot{H}^{-(2\beta+1),p'}},\end{aligned}$$

proving (6.53). Here, as $0 < -(2\beta+1) < 1$, the second inequality comes by interpolating the boundedness of $(e^{t\Delta})$ on $\dot{H}^{1,p'}$ (see Proposition 5.3) and that of $(t^{1/2}\nabla e^{t\Delta})$ on $L^{p'} \simeq \dot{H}^{0,p'}$. For $p = 2$, the same interpolation argument also yields

$$\|g\|_{T^2_{-(\beta+1/2)}} \eqsim \left(\int_0^t \|\nabla e^{(t-s)\Delta}\varphi\|^2_2 \, s^{2\beta+1} ds \right)^{1/2} \lesssim \|\varphi\|_{\dot{H}^{-(2\beta+1),2}}.$$

This shows (6.53). Then we apply the formula (b), duality of tent spaces, and the estimate (6.53) to get

$$\begin{aligned}|\langle u(t),\varphi\rangle| = |\langle \operatorname{div}\mathcal{L}_1^{-\Delta}(\widetilde{F})(t),\varphi\rangle| &\leq \int_0^t \int_{\mathbb{R}^n} |\widetilde{F}(s,x)||\nabla e^{-(t-s)\Delta}\varphi(x)|\,dsdx \\ &\lesssim \|\mathbb{1}_{(0,t)}\widetilde{F}\|_{T^p_{\beta+1/2}}\|g\|_{T^{p'}_{-(\beta+1/2)}} \lesssim \|\mathbb{1}_{(0,t)}\widetilde{F}\|_{T^p_{\beta+1/2}}\|\varphi\|_{\dot{H}^{-(2\beta+1),p'}}.\end{aligned}$$

Therefore, the claim (6.52) follows since for any $t>0$,

$$\|u(t)\|_{\dot{H}^{2\beta+1,p}} = \sup_{\varphi}|\langle u(t),\varphi\rangle| \lesssim \|\mathbb{1}_{(0,t)}\widetilde{F}\|_{T^p_{\beta+1/2}},$$

where the supremum is taken among $\varphi\in\mathscr{S}_\infty$ with $\|\varphi\|_{\dot{H}^{-(2\beta+1),p'}}=1$, noting that $\mathscr{S}_\infty$ is dense in $\dot{H}^{-(2\beta+1),p'}$.

Case 3: $-1<\beta<-1/2$ and $\widetilde{p}_L(\beta)<p\leq 1$ We use embedding of tent spaces. Pick $-1<\beta_0<\beta<-1/2$ and $\max\{\widetilde{p}_L(\beta_0),1\}<p_0<2$ as in Lemma 6.32 so that $T^p_{\beta+1/2}$ embeds into $T^{p_0}_{\beta_0+1/2}$. In particular, as $F\in T^p_{\beta+1/2}$, we have $F\in T^{p_0}_{\beta_0+1/2}$, for which we get back to Case 2. Hence, $u(t)$ tends to 0 as $t\to 0$ in $\dot{H}^{2\beta_0+1,p_0}$, and therefore in $\dot{H}^{-1,r}$ by Sobolev embedding and definition of r.

Case 4: $\beta\geq -1/2$ and $\widetilde{p}_L(\beta)<p\leq 2$ Caccioppoli's inequality (cf. Lemma 6.3) yields

$$\begin{aligned}\|u(t)\|_p^p &\leq \int_{\mathbb{R}^n}\left(\fint_{B(x,\frac{\sqrt{t}}{4})}|u(t,y)|^2\,dy\right)^{p/2}dx \\ &\lesssim \int_{\mathbb{R}^n}\left(\fint_{t/2}^t\fint_{B(x,\frac{\sqrt{t}}{2})}|u|^2\right)^{p/2}dx + \int_{\mathbb{R}^n}\left(\int_{t/2}^t\fint_{B(x,\frac{\sqrt{t}}{2})}|F|^2\right)^{p/2}dx.\end{aligned} \tag{6.54}$$

When $\beta=-1/2$, since $u\in T^p_{1/2}$ and $F\in T^p_0$, Lebesgue's dominated convergence theorem yields the integrals on the right-hand side tend to 0 as $t\to 0$. Thus, $u(t)$ tends to 0 in L^p as $t\to 0$. When $\beta>-1/2$, the right-hand side can be controlled by $t^{p(\beta+1/2)}(\|u\|_{T^p_{\beta+1}}+\|F\|_{T^p_{\beta+1/2}})^p$, which also tends to 0 as $t\to 0$ as $\beta>-1/2$.

It remains to prove that for $\beta\geq -1/2$ and $\widetilde{p}_L(\beta)<p<1$, $u(t)$ tends to 0 in $\mathscr{S}'$, because in this case, L^p does not embed into $\mathscr{S}'$. To this end, we also use embedding of tent spaces. Pick $\beta_0\in(-1,\beta]$ and $p_0>1$ as defined in Lemma 6.32, so $F\in T^{p_0}_{\beta_0+1/2}$. Then we apply the above discussion to get $u(t)$ tends to 0 as $t\to 0$ in L^{p_0} if $\beta_0\geq -1/2$, or in $\dot{H}^{-1,r}$ if $-1<\beta_0<-1/2$, see Case 2. But both spaces of course embed into $\mathscr{S}'$. We hence obtain the trace in $\mathscr{S}'$ as desired.

This completes the proof. □

6.4 Homogeneous Cauchy problem

This section is devoted to investigating existence of global weak solutions to the *homogeneous Cauchy problem*

$$\begin{cases} \partial_t v - \operatorname{div}(A\nabla v) = 0 \\ v(0) = v_0 \end{cases}, \tag{HC}$$

where v_0 lies in $\dot{H}^{2\beta+1,p}$ with $-1 < \beta < 0$.

Before stating our results, let us first briefly revisit the results for the heat equation in Section 5.2. For any $v_0 \in \mathscr{S}'$, the heat extension of v_0, $\mathcal{E}_{-\Delta}(v_0) \in C^\infty(\mathbb{R}^{1+n}_+)$ is a global weak (hence classical) solution to the heat equation with initial data v_0. We established a series of estimates on boundedness of the heat extension on $\dot{H}^{s,p}$, cf. Proposition 5.3, Theorem 5.4, and Corollary 5.6, with the correspondence $s = 2\beta + 1$.

For general L, we cannot start from tempered distributions. So for constructing weak solutions, our strategy is to extend the semigroup solution operator $\mathcal{E}_L$ defined in (6.5) initially on L^2 to $\dot{H}^{2\beta+1,p}$. Recall that it is given by

$$\mathcal{E}_L(v_0)(t) := e^{-tL} v_0, \quad t > 0.$$

When $L = -\Delta$, both approaches are consistent.

6.4.1 Main results

The following main theorem summarizes the properties of the extension of $\mathcal{E}_L$ to $\dot{H}^{2\beta+1,p}$.

Theorem 6.33 (Extension of $\mathcal{E}_L$)**.** *Let $-1 < \beta < 0$ and $\widetilde{p}_L(\beta) < p \le \infty$. Then $\mathcal{E}_L$ extends to an operator from $\dot{H}^{2\beta+1,p}$ to $L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$, also denoted by $\mathcal{E}_L$. Moreover, the following properties hold for any $v_0 \in \dot{H}^{2\beta+1,p}$ and $v := \mathcal{E}_L(v_0)$.*

(a) (Regularity) *∇v belongs to $T^p_{\beta+1/2}$ with equivalence*

$$\|\nabla v\|_{T^p_{\beta+1/2}} \eqsim \|v_0\|_{\dot{H}^{2\beta+1,p}}. \tag{6.55}$$

Moreover, if $-1 < \beta < -1/2$, then v lies in $T^p_{\beta+1}$ with

$$\|\nabla v\|_{T^p_{\beta+1/2}} \eqsim \|v\|_{T^p_{\beta+1}} \eqsim \|v_0\|_{\dot{H}^{2\beta+1,p}}. \tag{6.56}$$

(b) (Explicit formulae) *It holds that*

$$v = \mathcal{E}_{-\Delta}(v_0) + \mathcal{R}^L_{1/2}((A - \mathbb{I})\nabla \mathcal{E}_{-\Delta}(v_0)) \tag{6.57}$$

$$= \mathcal{E}_{-\Delta}(v_0) + \mathcal{R}^{-\Delta}_{1/2}((A - \mathbb{I})\nabla v), \tag{6.58}$$

where $\mathcal{E}_{-\Delta}$ is the heat extension defined in (5.3), and $\mathcal{R}^L_{1/2}, \mathcal{R}^{-\Delta}_{1/2}$ are given by Theorem 6.22.

(c) (Continuity) *v belongs to $C([0,\infty);\mathscr{S}')$ with $v(0)=v_0$.*

(d) (Strong continuity) *When $p_-(2\beta+1,L)<p<p_+(2\beta+1,L)$, v also belongs to $C^\infty((0,\infty);\dot{H}^{2\beta+1,p})\cap C_0([0,\infty);\dot{H}^{2\beta+1,p})$ with*

$$\sup_{t>0}\|v(t)\|_{\dot{H}^{2\beta+1,p}}\eqsim\|v_0\|_{\dot{H}^{2\beta+1,p}}.$$

(e) *v is a global weak solution to (HC) with initial data v_0.*

The proof is deferred to Section 6.4.4. Let us give some remarks.

Remark 6.34. When $\beta>-1/2$, the equivalence (6.56) fails: As we shall see in Theorem 6.44, any weak solution $v\in T^p_{\beta+1}$ to the equation $\partial_t v-\operatorname{div}(A\nabla v)=0$ must be zero when $p^\flat_L(\beta)<p\le\infty$, where $p^\flat_L(\beta)$ is given by (6.11).

For $\beta=-1/2$, it also fails. An alternative of the equivalence (6.56) can be achieved with the Kenig–Pipher space X^p as

$$\begin{cases}\|v\|_{X^p}\eqsim\|\nabla v\|_{T^p_0}\eqsim\|v_0\|_{L^p} & \text{if } p_-(L)=p_-(0,L)<p<\infty\\ \|v\|_{X^\infty}\eqsim\|v_0\|_{L^\infty}, & \text{if } p=\infty\\ \|\nabla v\|_{T^\infty_0}\eqsim\|v_0\|_{\mathrm{BMO}} & \text{if } p=\infty\end{cases}.$$

See [AMP19, Corollary 7.5, 5.5, & 5.10] and [Zat20, Theorem 7.6].

6.4.2 L-adapted Hardy–Sobolev spaces

Our main tools are L-adapted Hardy–Sobolev spaces. We give a brief review, and the reader can refer to [AA18, Chapter 4] and [AE23a, §8.2] for details. See also [HvNVW17, Chapter 10] for definitions and basic facts of sectorial operators and H^∞-calculus.

Denote by S^+_μ the set $\{z\in\mathbb{C}\setminus\{0\}:|\arg(z)|<\mu\}$ for $0<\mu<\pi$. The class $\Psi^\infty_\infty(S^+_\mu)$ consists of holomorphic functions $\psi:S^+_\mu\to\mathbb{C}$ fulfilling that for any $\sigma\in\mathbb{R}$, there exists a constant $C>0$ so that

$$|\psi(z)|\le C|z|^\sigma.$$

The class $H^\infty(S^+_\mu)$ consists of bounded holomorphic functions on S^+_μ.

It is well-known that L is a sectorial operator of angle ω for some $\omega\in[0,\pi/2)$ and has bounded $H^\infty(S^+_\mu)$-calculus for any $\mu\in(\omega,\pi)$. For any $\psi\in H^\infty(S^+_\mu)$, define the operator $\mathcal{Q}_{\psi,L}:L^2\to L^\infty((0,\infty);L^2)$ by

$$\mathcal{Q}_{\psi,L}(f)(t):=\psi(tL)f.$$

Note that for $\psi(z)=e^{-z}$, $\mathcal{Q}_{\psi,L}$ identifies with $\mathcal{E}_L$.

Definition 6.35 (L-adapted Hardy–Sobolev spaces)**.** Let $s \in \mathbb{R}$, $0 < p < \infty$, and $\omega < \mu < \pi$. Let ψ be a non-zero function in $H^\infty(S_\mu^+)$. The *L-adapted Hardy–Sobolev space* $\mathbb{H}^{s,p}_{L,\psi}$ consists of $f \in L^2$ so that $\mathcal{Q}_{\psi,L}(f)$ lies in $T^p_{(s+1)/2} \cap L^2_{1/2}(\mathbb{R}^{1+n}_+)$, equipped with the (quasi-)norm

$$\|f\|_{\mathbb{H}^{s,p}_{L,\psi}} := \|\mathcal{Q}_{\psi,L}(f)\|_{T^p_{(s+1)/2}}.$$

A priori, this space depends on the choice of ψ. However, for any (s,p) and $\psi \in \Psi^\infty_\infty(S_\mu^+)$, the spaces $\mathbb{H}^{s,p}_{L,\psi}$ agree as sets with mutually equivalent (quasi-)norms. We denote that space by $\mathbb{H}^{s,p}_L$. More generally, for each (s,p), there is a subclass of bounded holomorphic functions ψ containing Ψ^∞_∞, called *admissible functions for* $\mathbb{H}^{s,p}_L$, for which the spaces $\mathbb{H}^{s,p}_{L,\psi}$ are isomorphic to $\mathbb{H}^{s,p}_L$. See [AE23a, §8.2] for some explicit classes of admissible functions for $\mathbb{H}^{s,p}_L$, characterized by prescribed growth at origin and decay at infinity. For example, $\psi(z) = e^{-z}$ is admissible for $p \le 2$ and $s < 0$.

The spaces $\mathbb{H}^{s,p}_L$ are not complete for the defining (quasi-)norms. Aside from abstract completion, we are interested in realizing the completion as a subspace of tempered distributions. The next result provides a range of (s,p), called the identification range, within which a completion in fact equals to $\dot{H}^{s,p}$. When $0 \le s \le 1$, sharpness of this range is shown in [AE23a, §19.1]. We describe the range for $-1 \le s \le 0$, and also exhibit a possible extra range where a completion exists in $\dot{H}^{s,p}$ without knowing equality. This fact will be useful when we come to uniqueness.

We call *identification range* of L the set

$$\mathscr{I}_L := \{(s,p) \in [-1,1] \times (0,\infty) \,:\, p_-(s,L) < p < p_+(s,L)\}.$$

Proposition 6.36 (Completion of $\mathbb{H}^{s,p}_L$)**.** *Let $-1 \le s \le 1$ and $0 < p < \infty$.*

(i) (Identification) *If $(s,p) \in \mathscr{I}_L$, then $\mathbb{H}^{s,p}_L$ agrees with $\dot{H}^{s,p} \cap L^2$ with equivalent (quasi-)norms*

$$\|f\|_{\mathbb{H}^{s,p}_L} \eqsim \|f\|_{\dot{H}^{s,p}}.$$

In particular, $\dot{H}^{s,p}$ is a completion of $\mathbb{H}^{s,p}_L$.

(ii) (Extra embedding) *Suppose $p_-(L) < 1$. Let $-1 < s < 0$ and $1 < p < 2$. Then $\mathbb{H}^{s,p}_L \subset \dot{H}^{s,p}$ with*

$$\|f\|_{\dot{H}^{s,p}} \lesssim \|f\|_{\mathbb{H}^{s,p}_L}.$$

If, moreover,

$$s > -n\left(\frac{1}{p_-(L)} - 1\right) - \frac{q_+(L^*)}{p'}\left(1 - n\left(\frac{1}{p_-(L)} - 1\right)\right), \tag{6.59}$$

then the closure of $\mathbb{H}^{s,p}_L$ in $\dot{H}^{s,p}$ with respect to the norm $\|\cdot\|_{\mathbb{H}^{s,p}_L}$ is a completion of $\mathbb{H}^{s,p}_L$.

The proof is deferred to Section 6.4.5. Of course, the extra embedding range is only interesting when $(s,p) \notin \mathscr{I}_L$. In fact, setting $s = 2\beta + 1$, when $p_-(L) < 1$ and $\beta < \beta(L)$, the condition (6.59) exactly means $p > \widetilde{p}_L(\beta)$, so we observe from Figure 6.1 that such range can be non-empty.

Lemma 6.37 (Semigroup extension of $\mathcal{E}_L$)**.** *Let $s \in \mathbb{R}$ and $0 < p \leq \infty$. Then (e^{-tL}) is a continuous bounded semigroup on $\mathbb{H}^{s,p}_L$. In particular, for any $f \in \mathbb{H}^{s,p}_L$, $\mathcal{E}_L(f)$ lies in $C^\infty((0,\infty); \mathbb{H}^{s,p}_L) \cap C_0([0,\infty); \mathbb{H}^{s,p}_L)$ [3] with*

$$\sup_{t\geq 0} \|\mathcal{E}_L(f)(t)\|_{\mathbb{H}^{s,p}_L} = \sup_{t \geq 0} \|e^{-tL} f\|_{\mathbb{H}^{s,p}_L} \eqsim \|f\|_{\mathbb{H}^{s,p}_L}.$$

Restricting to the identification range $(s,p) \in \mathscr{I}_L$, (e^{-tL}) extends to a continuous bounded semigroup on $\dot{H}^{s,p}$, and for any $f \in \dot{H}^{s,p}$,

$$\mathcal{E}_L(f) \in C^\infty((0,\infty); \dot{H}^{s,p}) \cap C_0([0,\infty); \dot{H}^{s,p})$$

with

$$\sup_{t>0} \|\mathcal{E}_L(f)(t)\|_{\dot{H}^{s,p}} \eqsim \|f\|_{\dot{H}^{s,p}}.$$

Proof. The first point is a general result. See again [AE23a, §8.2] applied to $T = L$ and in particular, [AE23a, Proposition 8.13] replacing $\sqrt{T}$ by T. The second point follows from the first by Proposition 6.36 (i). $\square$

6.4.3 Distributional trace

Let us establish a general notion of distributional traces for weak solutions to the equation $\partial_t u - \operatorname{div}(A\nabla u) = 0$.

Proposition 6.38 (Trace)**.** *Let $\beta > -1$ and $\frac{n}{n+2\beta+2} < p \leq \infty$. Let u be a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = 0$ with $\nabla u \in T^p_{\beta+1/2}$. Then there exists a unique $u_0 \in \mathscr{S}'$ such that $u(t)$ converges to u_0 in $\mathscr{S}'$ as $t \to 0$, and*

$$u = \mathcal{E}_{-\Delta}(u_0) + \mathcal{R}^{-\Delta}_{1/2}((A - \mathbb{I})\nabla u). \tag{6.60}$$

In addition,

(i) *If $\beta \geq 0$ and $\frac{n}{n+2\beta} \leq p \leq \infty$, then u_0 is a constant.*

(ii) *If $-1 < \beta < 0$, then there exist $g \in \dot{H}^{2\beta+1,p}$ and $c \in \mathbb{C}$ so that $u_0 = g + c$ with*

$$\|g\|_{\dot{H}^{2\beta+1,p}} \lesssim \|\nabla u\|_{T^p_{\beta+1/2}}.$$

[3] For $p = \infty$, $\mathbb{H}^{s,\infty}_L$ is contained in the dual of $\mathbb{H}^{-s,1}_{L^*}$ by $L^2(\mathbb{R}^n)$-inner product. Here, it is equipped with the induced weak*-topology.

Proof. Note that $\frac{n}{n+2\beta+2} = \widetilde{p}_{-\Delta}(\beta)$, so by Theorem 6.22 (a), as $\nabla u \in T^p_{\beta+1/2}$, we get $\mathcal{R}^{-\Delta}_{1/2}((A-\mathbb{I})\nabla u) \in T^p_{\beta+1}$ and $\nabla \mathcal{R}^{-\Delta}_{1/2}((A-\mathbb{I})\nabla u) \in T^p_{\beta+1/2}$. Besides, Theorem 6.22 (d) says that $\mathcal{R}^{-\Delta}_{1/2}((A-\mathbb{I})\nabla u)(t)$ vanishes in $\mathscr{S}'$ as $t \to 0$.

Define $\widetilde{u} \in L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$ by

$$\widetilde{u} := u - \mathcal{R}^{-\Delta}_{1/2}((A-\mathbb{I})\nabla u).$$

Observe that $\widetilde{u}$ is a global weak (hence distributional) solution to the heat equation with $\nabla \widetilde{u} \in T^p_{\beta+1/2}$. Then we invoke Theorem 5.17 to get $u_0 \in \mathscr{S}'$ so that $\widetilde{u} = \mathcal{E}_{-\Delta}(u_0)$, which proves (6.60) and also implies that $u(t)$ converges to u_0 in $\mathscr{S}'$ as $t \to 0$. Moreover, Theorem 5.17 (ii) says that u_0 has the properties asserted in (i) and (ii). This completes the proof. □

6.4.4 Proof of Theorem 6.33

We use the formula of $\mathcal{E}_L$ shown in Corollary 6.7 (ii) to construct its extension. Define the operator $\mathcal{T}_{1/2}$ from L^2 to $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$ by

$$\mathcal{T}_{1/2}(f) = \mathcal{R}^L_{1/2}((A-\mathbb{I})\nabla \mathcal{E}_{-\Delta}(f)). \tag{6.61}$$

We first extend $\mathcal{T}_{1/2}$ to $\dot{H}^{2\beta+1,p}$.

Lemma 6.39 (Extension of $\mathcal{T}_{1/2}$). *Let $-1 < \beta < 0$ and $\widetilde{p}_L(\beta) < p \leq \infty$. Then for any $f \in \dot{H}^{2\beta+1,p} \cap L^2$,*

$$\|\mathcal{T}_{1/2}(f)\|_{T^p_{\beta+1}} + \|\nabla \mathcal{T}_{1/2}(f)\|_{T^p_{\beta+1/2}} \lesssim \|f\|_{\dot{H}^{2\beta+1,p}}.$$

Hence, $\mathcal{T}_{1/2}$ extends by density (or weak-density if $p = \infty$) to a bounded operator from $\dot{H}^{2\beta+1,p}$ to $T^p_{\beta+1}$ with the above estimate. We use the same notation for the extension.*

Proof. We just need to show the inequality. To this end, fix $f \in \dot{H}^{2\beta+1,p} \cap L^2$. The formula (6.61), together with estimates from Theorem 6.22 (a) applied to $\mathcal{R}^L_{1/2}$, yields that

$$\|\mathcal{T}_{1/2}(f)\|_{T^p_{\beta+1}} + \|\nabla \mathcal{T}_{1/2}(f)\|_{T^p_{\beta+1/2}} \lesssim \|\nabla \mathcal{E}_{-\Delta}(f)\|_{T^p_{\beta+1/2}} \lesssim \|f\|_{\dot{H}^{2\beta+1,p}}.$$

The last inequality follows from Corollary 5.6 (i). □

Proof of Theorem 6.33. Let $-1 < \beta < 0$ and $\widetilde{p}_L(\beta) < p \leq \infty$. We define the extension $\mathcal{E}_L$ by

$$\mathcal{E}_L(v_0) := \mathcal{E}_{-\Delta}(v_0) + \mathcal{T}_{1/2}(v_0), \quad \forall v_0 \in \dot{H}^{2\beta+1,p}, \tag{6.62}$$

where $\mathcal{T}_{1/2}$ is given by Lemma 6.39.

Let us verify the properties for $v := \mathcal{E}_L(v_0)$. As $\dot{H}^{2\beta+1,p}$ is contained in $\mathscr{S}'$, we get $\mathcal{E}_{-\Delta}(v_0) \in C^\infty(\mathbb{R}^{1+n}_+)$. Also, Lemma 6.39 says $\mathcal{T}_{1/2}(v_0) \in T^p_{\beta+1}$ and $\nabla\mathcal{T}_{1/2}(v_0) \in T^p_{\beta+1/2}$. In particular, it implies $\mathcal{T}_{1/2}(v_0) \in L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$, and so does v.

First consider (a). The inequality "$\lesssim$" in (6.55) follows directly from Corollary 5.6 (i) and Lemma 6.39 as

$$\|\nabla v\|_{T^p_{\beta+1/2}} \leq \|\nabla \mathcal{E}_{-\Delta}(v_0)\|_{T^p_{\beta+1/2}} + \|\nabla\mathcal{T}_{1/2}(v_0)\|_{T^p_{\beta+1/2}} \lesssim \|v_0\|_{\dot{H}^{2\beta+1,p}}.$$

We postpone the proof of the converse inequality "$\gtrsim$" in (6.55) and (6.56) to the end of the proof.

Next, the formulae in (b) hold when $v_0 \in \dot{H}^{2\beta+1,p} \cap L^2$ by Corollary 6.7, and the above bounds of the terms involved allow us to use density (or weak*-density) to extend them to all $v_0 \in \dot{H}^{s,p}$, valued in $L^2_{\mathsf{loc}}(\mathbb{R}^{n+1}_+)$.

To prove (c), we use the formula (6.58) and prove that each term has the desired regularity. As $\dot{H}^{2\beta+1,p}$ is contained in $\mathscr{S}'$, we have $\mathcal{E}_{-\Delta}(v_0) \in C([0,\infty);\mathscr{S}')$. Moreover, as $\widetilde{p}_{-\Delta}(\beta) \leq \widetilde{p}_L(\beta) < p$ and $\nabla v \in T^p_{\beta+1/2}$, we deduce from Theorem 6.22 (d) that $\mathcal{R}^{-\Delta}_{1/2}((A-\mathbb{I})\nabla v) \in C([0,\infty);\mathscr{S}')$ as wanted.

Property (d) is a direct consequence of uniqueness of extensions by density: the extension defined by (6.62) agrees with the one given by Lemma 6.37, which has the desired regularity properties.

For (e), note that $\mathcal{E}_{-\Delta}(v_0)$ is a global weak solution to the heat equation, and Theorem 6.22 yields that $w := \mathcal{R}^{-\Delta}_{1/2}((A-\mathbb{I})\nabla v)$ is a global weak solution to $\partial_t w - \Delta w = \operatorname{div}((A-\mathbb{I})\nabla v)$. Therefore, we infer from formula (6.58) that v is a global weak solution to $\partial_t v - \Delta v = \operatorname{div}((A-\mathbb{I})\nabla v)$, that is $\partial_t v - \operatorname{div}(A\nabla v) = 0$. Meanwhile, (c) shows that $v(t)$ converges to v_0 in $\mathscr{S}'$ as $t \to 0$. We hence conclude that v is a global weak solution to (HC) with initial data v_0.

Let us now prove the rest of (a). We begin with the inequality "$\gtrsim$" in (6.55). We just proved in (e) that v is a global weak solution to the equation $\partial_t v - \operatorname{div}(A\nabla v) = 0$ with $\nabla v \in T^p_{\beta+1/2}$, and that $v(t)$ converges to v_0 in $\mathscr{S}'$ as $t \to 0$, so v_0 agrees with the trace g of v given by Proposition 6.38 (ii) in $\dot{H}^{2\beta+1,p}$. In particular, we get $\|v_0\|_{\dot{H}^{2\beta+1,p}} \lesssim \|\nabla v\|_{T^p_{\beta+1/2}}$ as desired. This proves (6.55).

Next, to see (6.56), we only need to show that $\|v\|_{T^p_{\beta+1}} \lesssim \|v_0\|_{\dot{H}^{2\beta+1,p}}$, thanks to (6.55) and Corollary 6.4. This inequality is a direct consequence of (6.58), Theorem 5.4 (i), and (6.55) (for the Laplacian) as

$$\begin{aligned}\|v\|_{T^p_{\beta+1}} &\lesssim \|\mathcal{E}_{-\Delta}(v_0)\|_{T^p_{\beta+1}} + \|\mathcal{R}^{-\Delta}_{1/2}((A-\mathbb{I})\nabla v)\|_{T^p_{\beta+1}} \\ &\lesssim \|\mathcal{E}_{-\Delta}(v_0)\|_{T^p_{\beta+1}} + \|\nabla v\|_{T^p_{\beta+1/2}} \lesssim \|v_0\|_{\dot{H}^{2\beta+1,p}}.\end{aligned}$$

This shows (a) and the proof is complete. □

6.4.5 Proof of Proposition 6.36

6.4.5.1 Proof of Proposition 6.36 (i)

Let us first establish an appropriate atomic decomposition for distributions in $\dot{H}^{1,p}$ with $\frac{n}{n+2} < p \leq \frac{n}{n+1}$. For $\frac{n}{n+1} < p \leq 1$, it was done by [AE23a, Proposition 8.31].

Definition 6.40 ($\dot{H}^{1,p}$-atom)**.** Let $\frac{n}{n+2} < p \leq \frac{n}{n+1}$. A function $a \in L^2$ is called an *$\dot{H}^{1,p}$-atom* if

(i) there is a ball $B \subset \mathbb{R}^n$ so that $\operatorname{supp}(a) \subset B$, which is called *associated* to a;

(ii) a is of mean zero, *i.e.*, $\int_{\mathbb{R}^n} a = 0$;

(iii) $\|\nabla a\|_2 \leq |B|^{[2,p]}$.

Compared with the case for $\frac{n}{n+1} < p \leq 1$ (see [AE23a, Definition 8.30]), the only extra condition we impose is (ii).

Lemma 6.41 (Atomic decomposition for $\dot{H}^{1,p}$)**.** *Let $\frac{n}{n+2} < p \leq \frac{n}{n+1}$. For any $f \in \dot{H}^{1,p} \cap W^{1,2}$, there exist $(\lambda_i) \in \ell^p$ and $\dot{H}^{1,p}$-atoms a_i so that*

$$f = \sum_{i \geq 1} \lambda_i a_i,$$

where the convergence holds in $\dot{W}^{1,2}$ (that is, with respect to the semi-norm $\|\nabla \cdot \|_2$). Moreover, the estimate holds as

$$\|f\|_{\dot{H}^{1,p}} \lesssim \|(\lambda_i)\|_{\ell^p}.$$

Proof. The proof is analogous to that of [AE23a, Proposition 8.31]. We only give a sketch here. Let D be the *Dirac operator*, which is defined on $L^2(\mathbb{R}^n; \mathbb{C}^{1+n})$ by

$$D := \begin{bmatrix} 0 & \operatorname{div} \\ -\nabla & 0 \end{bmatrix}.$$

The main strategy of their proof is to use the atomic decomposition of D-adapted Hardy space $\mathbb{H}^p_D$ $(0 < p < \infty)$ to the vector $g := D[f, 0]^T = -[0, \nabla f]^T$. The only difference is that we decompose g with $(\mathbb{H}^p_D, 1, 2)$-molecules instead of $(\mathbb{H}^p_D, 1, 1)$-molecules, which is allowed by [AE23a, Theorem 8.17]. The same localization arguments apply to obtain an L^2-convergent decomposition $\nabla f = \sum_{i \geq 1} \lambda_i \nabla a_i$, with λ_i, a_i as required. $\square$

For any $p \in (0, \infty]$, denote by p_* the lower Sobolev conjugate of p, *i.e.*, $1/p_* := 1/p + 1/n$. For any $p \in (0, n)$, denote by p^* the upper Sobolev conjugate of p, *i.e.*, $1/p^* := 1/p - 1/n$.

Proof of Proposition 6.36 (i). We only need to show $\mathbb{H}^{s,p}_L$ agrees with $\dot{H}^{s,p} \cap L^2$ with equivalent (quasi-)norms

$$\|f\|_{\mathbb{H}^{s,p}_L} \eqsim \|f\|_{\dot{H}^{s,p}},$$

when $-1 \le s \le 1$ and $p_-(s,L) < p < p_+(s,L)$. First observe that by interpolation of $\mathbb{H}^{s,p}_L$ (see [AA18, Theorem 4.28]), it suffices to consider three cases, $s=0$ and $s=\pm 1$.

For $s=0$, this is [AE23a, Theorem 9.7], as $p_\pm(0,L) = p_\pm(L)$.

For $s=1$, [AE23a, Theorem 9.7] shows that it holds when $(\max\{p_-(L),1\})_* < p < q_+(L) = p_+(1,L)$, so it remains to consider the case when $\frac{n}{n+1} \le p_-(L) < 1$ and $p_-(1,L) = (p_-(L))_* < p \le 1_*$. The method of proof is adapted from the argument for [AE23a, §9.2, Part 5], and we only need to show that for any $\psi \in \Psi^\infty_\infty$, there is a constant $C>0$ so that for any $\dot{H}^{1,p}$-atom a,

$$\|\psi(tL)a\|_{T^p_1} \le C. \tag{6.63}$$

Let us verify it. Let ψ be in Ψ^∞_∞, a be an $\dot{H}^{1,p}$-atom, and B be a ball of $\mathbb{R}^n$ associated to a. Note that

$$\|\psi(tL)a\|_{T^p_1} \lesssim \|\mathcal{A}(t^{-1}\psi(tL)a)\|_{L^p(4B)} + \|\mathcal{A}(t^{-1}\psi(tL)a)\|_{L^p((4B)^c)} =: I_1 + I_2,$$

where $\mathcal{A}$ is the conical square function defined by

$$\mathcal{A}(f)(x) := \left(\int_0^\infty \fint_{B(x,t^{1/2})} |f(t,y)|^2 dt dy \right)^{1/2}.$$

It coincides with $\mathcal{A}^{(1)}_{0;2}(f)(x)$ defined in (3.1). The boundedness of I_1 is exactly as for [AE23a, (9.30)]. Let us concentrate on I_2. To this end, we fix $(t,x) \in \mathbb{R}^{1+n}_+$ with $x \in (4B)^c$. As a is of mean zero, Poincaré's inequality yields

$$\|a\|_2 \lesssim r(B)\|\nabla a\|_2 \lesssim |B|^{[2,p^*]}. \tag{6.64}$$

This estimate, together with the support condition and mean value 0, shows that a belongs to H^q for $\frac{n}{n+1} < q \le 1$ with the estimate

$$\|a\|_{H^q} \lesssim |B|^{[q,p^*]}. \tag{6.65}$$

Then we fix $p_-(L) < q < p^*$. The $H^q - L^2$ boundedness of $(\psi(tL))_{t>0}$ (see [AE23a, Lemma 4.4]) yields

$$\|\psi(tL)a\|_2 \lesssim t^{\frac{n}{2}[2,q]} \|a\|_{H^q}. \tag{6.66}$$

Meanwhile, using $L^2 - L^2$ off-diagonal estimates, [4] we get that for any $M>0$,

$$\|\psi(tL)a\|_{L^2(B(x,t^{1/2}))} \lesssim \left(1 + \frac{\operatorname{dist}(B(x,t^{1/2}),B)^2}{t}\right)^{-M} \|a\|_2. \tag{6.67}$$

[4] Here we use polynomial-order decay as in [AE23a, Lemma 4.16].

As $q < p^* < 2$, *i.e.*, $n/q + 1 > n/p > n/2$, we pick $\theta \in (0,1)$ so that

$$\rho := \frac{n}{q} + 1 - n\theta[q,2] > \frac{n}{p}. \tag{6.68}$$

Note that θ can be chosen arbitrarily close to 0. We interpolate (6.66) and (6.67) with respect to θ to get

$$\begin{aligned}&\|\psi(tL)a\|_{L^2(B(x,t^{1/2}))}\\ &\quad\lesssim t^{\frac{n}{2}[2,q](1-\theta)}\left(1+\frac{\operatorname{dist}(B(x,t^{1/2}),B)^2}{t}\right)^{-M\theta}\|a\|_{H^q}^{1-\theta}\|a\|_2^{\theta}.\end{aligned}$$

Taking $M > \frac{\rho}{2\theta}$, we integrate it over t (with right weight and norm) to get

$$\mathcal{A}(t^{-1}\psi(tL)a)(x) \lesssim \operatorname{dist}(x,B)^{-\rho}\|a\|_{H^q}^{1-\theta}\|a\|_2^{\theta}.$$

Gathering (6.64), (6.65), and definition of $\rho > n/p$ (cf. (6.68)), we obtain

$$\begin{aligned}I_2 = \|\mathcal{A}(t^{-1}\psi(tL)a)\|_{L^p((4B)^c)} &\lesssim r(B)^{\frac{n}{p}-\rho}\|a\|_{H^q}^{1-\theta}\|a\|_2^{\theta}\\ &\lesssim |B|^{\frac{1}{p}-\frac{1}{n}-\frac{1}{q}+\theta[q,2]}|B|^{(1-\theta)[q,p^*]}|B|^{\theta[2,p^*]} \lesssim 1.\end{aligned}$$

This proves (6.63) and hence concludes the case $s = 1$.

For $s = -1$, we use duality. As $1 \le p_-(-1,L) < p < p_+(-1,L) \le \infty$, we have $\max\{p_-(1,L^*),1\} < p' < p_+(1,L^*) = q_+(L^*)$. Then for any $f \in \mathbb{H}_L^{-1,p}$, we have

$$\|f\|_{\dot{H}^{-1,p}} = \sup_{g\in \dot{H}^{1,p'}\cap L^2(\mathbb{R}^n)} \frac{|\langle f,g\rangle|}{\|g\|_{\dot{H}^{1,p'}}} \eqsim \sup_{g\in\mathbb{H}_{L^*}^{1,p'}} \frac{|\langle f,g\rangle|}{\|g\|_{\mathbb{H}_{L^*}^{1,p'}}} \eqsim \|f\|_{\mathbb{H}_L^{-1,p}}.$$

The equality follows by density of $\dot{H}^{1,p'}\cap L^2$ in $\dot{H}^{1,p'}$ and the first equivalence by $\mathbb{H}_{L^*}^{1,p'} = \dot{H}^{1,p'}\cap L^2$ with $\|g\|_{\mathbb{H}_{L^*}^{1,p'}} \eqsim \|g\|_{\dot{H}^{1,p'}}$ (from Case $s = 1$ for L^*). The last equivalence holds by [AE23a, Proposition 8.9]. This completes the proof. □

6.4.5.2 Proof of Proposition 6.36 (ii)

Recall that we assume $p_-(L) < 1$, which implies $p_+(L^*) = \infty$. When $1 < p < 2$ and $-1 \le s \le 0$, by duality, it is equivalent to prove $\dot{H}^{-s,p'}\cap L^2 \subset \mathbb{H}_{L^*}^{-s,p'}$ with $\|f\|_{\mathbb{H}_{L^*}^{-s,p'}} \lesssim \|f\|_{\dot{H}^{-s,p'}}$. For $s = 0$ and $s = -1$, this is explicitly proved in Part 2 and Part 9 of the proof of [AE23a, Theorem 9.7]. Interpolation concludes the argument.

It remains to show that the extension of the identity map to the closure of $\mathbb{H}_L^{s,p}$ (for its norm) is injective in the restricted range described in the statement. More precisely, let (f_j) be a Cauchy sequence in $\mathbb{H}_L^{s,p}$ that tends to 0 in $\dot{H}^{s,p}$. Our goal is to show $\|f_j\|_{\mathbb{H}_L^{s,p}}$ tends to 0.

To this end, we use duality. Pick $\phi \in C_c^\infty(\mathbb{R}^{1+n}_+)$ and define

$$\Phi := \int_0^\infty e^{-tL^*}\phi(t)\frac{dt}{t}.$$

Fubini's theorem ensures

$$\int_0^\infty \int_{\mathbb{R}^n} (e^{-tL}f_j)(y)\overline{\phi}(t,y)dy\frac{dt}{t} = \int_{\mathbb{R}^n} f_j(y)\overline{\Phi(y)}dy.$$

We claim $\Phi \in \dot{H}^{-s,p'}$. If so, then using the fact that (f_j) tends to 0 in $\dot{H}^{s,p}$, we get $(\mathcal{E}_L(f_j))$ tends to 0 in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$ by arbitrariness of ϕ. Moreover, as (f_j) is a Cauchy sequence in $\mathbb{H}^{s,p}_L$, so is $(\mathcal{E}_L(f_j))$ in $T^p_{(s+1)/2}$, because the exponential function e^{-z} is an admissible function for $\mathbb{H}^{s,p}_L$ when $s<0$ and $p \le 2$. Hence, it tends to 0 in $T^p_{(s+1)/2}$, which implies $\lim_{j\to\infty}\|f_j\|_{\mathbb{H}^{s,p}_L} = 0$.

We finish by verifying the claim. Pick $0<a<b<\infty$ so that $\operatorname{supp}(\phi) \subset (a,b)\times\mathbb{R}^n$. Note that $\phi \in C_c^\infty(\mathbb{R}^{1+n}_+)$ implies $\Phi \in L^2$, so we apply Theorem 5.4 to $\Delta\Phi$ to get

$$\|\Phi\|_{\dot{H}^{-s,p'}} \eqsim \|\tau\Delta e^{\tau\Delta}\Phi\|_{T^{p'}_{(-s+1)/2}}. \tag{6.69}$$

Pick $0<a<b<\infty$ so that $\operatorname{supp}(\phi)\subset(a,b)\times\mathbb{R}^n$. Using Hölder's inequality and Minkowski's inequality, we have (the implicit constants may depend on a,b)

$$\begin{aligned}
&\|\tau\Delta e^{\tau\Delta}\Phi\|_{T^{p'}_{(-s+1)/2}} \\
&\lesssim \left(\int_{\mathbb{R}^n}\left(\int_0^\infty \fint_{B(x,\tau^{1/2})}\int_a^b |\tau^{\frac{s-1}{2}}\tau\Delta e^{\tau\Delta}e^{-tL^*}\phi(t)|^2\,dtdyd\tau\right)^{p'/2} dx\right)^{1/p'} \\
&\lesssim \left(\int_a^b\int_0^\infty\left(\int_{\mathbb{R}^n}\left(\fint_{B(x,\tau^{1/2})}|\tau^{\frac{s-1}{2}}\tau\Delta e^{\tau\Delta}e^{-tL^*}\phi(t)|^2\,dy\right)^{\frac{p'}{2}}dx\right)^{\frac{2}{p'}}d\tau dt\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_a^b\int_0^\infty \|\tau^{\frac{s-1}{2}}\tau\Delta e^{\tau\Delta}e^{-tL^*}\phi(t)\|_{p'}^2\,d\tau dt\right)^{1/2}
\end{aligned}$$

When τ is large, uniform $L^{p'}$-boundedness of $(\tau\Delta e^{\tau\Delta})$ and (e^{-tL^*}) implies

$$\|\tau\Delta e^{\tau\Delta}e^{-tL^*}\phi(t)\|_{p'} \lesssim \|\phi(t)\|_{p'}.$$

As $s<0$, $s-1<-1$, so the convergence of the integral when $\tau\ge 1$ is ensured.

When τ is small, we need different methods to gain a positive power of τ. For $2\le q<q_+(L^*)'$, we use $L^{q'}$-boundedness of $(e^{\tau\Delta}\operatorname{div})$ and $(t^{1/2}\nabla e^{-tL^*})$ for the decomposition to get

$$\begin{aligned}
\|\tau\Delta e^{\tau\Delta}e^{-tL^*}\phi(t)\|_q &= \tau^{1/2}t^{-1/2}\left\|\left(\tau^{1/2}e^{\tau\Delta}\operatorname{div}\right)\left(t^{1/2}\nabla e^{-tL^*}\phi(t)\right)\right\|_q \\
&\lesssim \tau^{1/2}t^{-1/2}\|\phi(t)\|_q.
\end{aligned} \tag{6.70}$$

Meanwhile, let $\alpha \in (0,1)$ be a parameter to be determined with the constraint

$$0 < \alpha < n\left(\frac{1}{p_-(L)} - 1\right).$$

Denote by $\dot{\Lambda}^\alpha$ the homogeneous α-Hölder space. One can also use Gaussian decay of the kernel of $(\tau\Delta e^{\tau\Delta})$ and uniform $\dot{\Lambda}^\alpha$-boundedness of (e^{-tL^*}) (by duality from H^p-boundedness of (e^{-tL})) to get

$$\|\tau\Delta e^{\tau\Delta} e^{-tL^*}\phi(t)\|_\infty \lesssim \tau^{\alpha/2}\|e^{-tL^*}\phi(t)\|_{\dot{\Lambda}^\alpha} \lesssim \tau^{\alpha/2}\|\phi(t)\|_{\dot{\Lambda}^\alpha}. \tag{6.71}$$

Interpolating (6.70) and (6.71) yields

$$\|\tau\Delta e^{\tau\Delta} e^{-tL^*}\phi(t)\|_{p'} \lesssim_\phi \tau^{\frac{\alpha}{2}(1-\frac{q}{p'})+\frac{q}{2p'}}. \tag{6.72}$$

Observe that when $s > -n(\frac{1}{p_-(L)}-1) - \frac{q_+(L^*)}{p'}(1-n(\frac{1}{p_-(L)}-1))$, *i.e.*, (6.59) is satisfied, there exist $\alpha \in (0,1)$ and $q \in [2, q_+(L^*))$ so that

$$\begin{cases} 0 < \alpha < n(\frac{1}{p_-(L)} - 1) \\ s - 1 + \alpha(1-\frac{q}{p'}) + \frac{q}{p'} > -1 \end{cases}.$$

So the convergence of the integral for small τ follows. We thus obtain $\Phi \in \dot{H}^{-s,p'}$ from (6.69). This completes the proof.

6.5 Uniqueness and representation

This section is devoted to prove uniqueness and representation of weak solutions in the solution to (6.1) in the solution class $\nabla u \in T^p_{\beta+1/2}$.

6.5.1 Main results

Theorem 6.42 (Uniqueness)**.** *Let $\beta > -1$ and $\widetilde{p}_L(\beta) < p \le \infty$. Let u be a global weak solution to the initial value problem*

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = 0, \\ u(0) = 0 \end{cases},$$

with $\nabla u \in T^p_{\beta+1/2}$. Then $u = 0$.

Theorem 6.43 (Representation)**.** *Let $\beta > -1$ and $\widetilde{p}_L(\beta) < p \le \infty$. Let u be a weak solution to the equation*

$$\partial_t u - \operatorname{div}(A\nabla u) = 0, \quad (t,x) \in (0,\infty) \times \mathbb{R}^n,$$

with $\nabla u \in T^p_{\beta+1/2}$. Then u has a trace $u_0 \in \mathscr{S}'$ in the sense that $u(t)$ converges to u_0 in $\mathscr{S}'$ as $t \to 0$. Moreover,

(i) *If* $\beta \geq 0$ *and* $\frac{n}{n+2\beta} \leq p \leq \infty$, *then* u *is a constant.*

(ii) *If* $-1 < \beta < 0$, *then there exist* $g \in \dot{H}^{2\beta+1,p}$ *and* $c \in \mathbb{C}$ *so that* $u_0 = g + c$ *and* $u = \mathcal{E}_L(g) + c$, *where* $\mathcal{E}_L$ *is given by Theorem 6.33.*

The proofs of Theorems 6.42 and 6.43 are postponed to Sections 6.5.4 and 6.5.5, respectively. Let us introduce another uniqueness class, for which no initial condition needs to be imposed.

Theorem 6.44. *Let* $\beta > -1/2$ *and* $p_L^\flat(\beta) < p \leq \infty$. *Then, for any* $f \in T^p_\beta$, *there exists at most one global weak solution* $u \in T^p_{\beta+1}$ *to the equation* $\partial_t u - \operatorname{div}(A\nabla u) = f$.

The proof is presented in Section 6.5.3. We first make some remarks.

Remark 6.45. Let us first clarify the absence of initial conditions in Theorem 6.44. Recall that in Lemma 6.18, we establish another notion of trace, so-called *(parabolic) Whitney trace*, for all the functions (not only solutions) in the space $T^p_{\beta+1}$ for $0 < p \leq \infty$ and $\beta > -1/2$ (or $s = 2\beta + 1 > 0$). This notion differs from the usual notion by taking completion from the usual trace of test functions. It consists in taking the limit of averages on parabolic Whitney cubes as $t \to 0$ for a.e. $x \in \mathbb{R}^n$. In particular, the solution in the statement must have zero Whitney trace. It implies that there is no non-trivial global weak solution in $T^p_{\beta+1}$ of $\partial_t u - \operatorname{div}(A\nabla u) = 0$, and hence the initial value problem cannot be posed in $T^p_{\beta+1}$ when $\beta > -1/2$.

Remark 6.46. Let us compare Theorems 6.42 and 6.44. Recall that Corollary 6.4 asserts the inequality $\|\nabla v\|_{T^p_{\beta+1/2}} \lesssim \|v\|_{T^p_{\beta+1}}$ holds for any global weak solution v to the equation $\partial_t v - \operatorname{div}(A\nabla v) = 0$. Therefore, we improve the class of uniqueness in this range of β with more values of p, with the price of imposing the initial condition.

6.5.2 Homotopy identity

In the spirit of the heat equation (see Section 5.3), our strategy is to employ the interior representation of weak solutions, so-called "homotopy identity", to reduce the problem of uniqueness to understanding the boundary behavior of solutions.

Let us first recall the homotopy identity for general parabolic equations with rough coefficients, which was first proved in [AMP19, Theorem 5.1] for time-dependent coefficients.

Proposition 6.47 (Homotopy identity)**.** *Let* $0 \leq a < b \leq \infty$. *Let* $u \in L^2_{\mathsf{loc}}((a,b); W^{1,2}_{\mathsf{loc}}(\mathbb{R}^n))$ *be a weak solution to the equation* $\partial_t u - \operatorname{div}(A\nabla u) = 0$ *on* $(a,b) \times \mathbb{R}^n$. *Suppose that there exists*

$$0 < \gamma < \frac{\Lambda_0}{16\Lambda_1^2(b-a)}$$

so that

$$\int_{\mathbb{R}^n} \left(\int_a^b \int_{B(x,b^{1/2})} |u(t,y)|^2 dt dy \right)^{1/2} e^{-\gamma |x|^2} dx < \infty.$$

Then when $a < s < t < b$, for any $h \in C_c^\infty(\mathbb{R}^n)$, it holds that

$$\int_{\mathbb{R}^n} u(t,x)\overline{h}(x)dx = \int_{\mathbb{R}^n} u(s,x)\overline{(e^{-(t-s)L^*}h)}(x)dx.$$

Next, the following standard corollary of Banach–Steinhaus theorem allows us to study separately the convergence of $u(s)$ and $e^{-(t-s)L^*}h$ as $s \to 0$.

Lemma 6.48. *Let X be a Banach space and X^* be its dual. Let (x_j) be a sequence in X that converges to x in X and (y_j) be a sequence in X^* that weakly* converges to y in X^*. Then, the sequence of pairing $(\langle y_j, x_j \rangle)$ converges to $\langle y, x \rangle$.*

6.5.3 Proof of Theorem 6.44

We first demonstrate Theorem 6.44. Let us prepare a lemma on the continuity of the semigroup $(e^{-tL})_{t \geq 0}$ in slice spaces E^p_δ (see Section 3.4.1 for the definition).

Lemma 6.49. *Let $1 \leq p < \infty$ and $\delta > 0$. Then the semigroup (e^{-tL}) is strongly continuous on E^p_δ.*

Proof. It is a direct consequence of [AM19, Proposition 4.4], thanks to the fact that the semigroup (e^{-tL}) is analytic and has $L^2 - L^2$ off-diagonal estimates of arbitrary order. □

We present the proof of Theorem 6.44.

Proof of Theorem 6.44. Let u and $\widetilde{u}$ be two global weak solutions in $T^p_{\beta+1}$ to the equation $\partial_t u - \operatorname{div}(A\nabla u) = f$ with same source term $f \in T^p_\beta$.

Define $v := u - \widetilde{u}$. It belongs to $L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}}) \cap T^p_{\beta+1}$. Moreover, it is a global weak solution to the equation $\partial_t v - \operatorname{div}(A\nabla v) = 0$. For $0 < a < b < \infty$ and $\gamma > 0$, we have

$$\int_{\mathbb{R}^n} \left(\int_a^b \int_{B(x,b^{1/2})} |v(t,y)|^2 dt dy \right)^{1/2} e^{-\gamma |x|^2} dx \lesssim_{\gamma,p,a,b,\beta} \|v\|_{T^p_\beta} < \infty,$$

thanks to Lemma 5.20 for $p^\flat_L(\beta) < p \leq 2$ and Lemma 5.21 for $2 \leq p \leq \infty$. Thus, Proposition 6.47 ensures that the homotopy identity holds: For $0 < s < t < \infty$ and $h \in C_c^\infty(\mathbb{R}^n)$, we have

$$\int_{\mathbb{R}^n} v(t,x)\overline{h}(x)dx = \int_{\mathbb{R}^n} v(s,x)\overline{((e^{-(t-s)L})^*h)}(x)dx. \tag{6.73}$$

From now on, fix $t > 0$. We wish to take limits as $s \to 0$ on both sides in (6.73) to show $v(t) = 0$ in $\mathscr{D}'(\mathbb{R}^n)$, hence proving $v \equiv 0$. The proof is divided into three cases.

Case 1: $2 \le p < \infty$ Let $\delta > 0$ be a constant so that $0 < s < t < \delta$. Since v is a global weak solution to $\partial_t v - \operatorname{div}(A\nabla v) = 0$, Caccioppoli's inequality (cf. Lemma 6.3) yields

$$\|v(s)\|_{E^p_{s/16}} = \left(\int_{\mathbb{R}^n} \left(\fint_{B(x,\frac{\sqrt{s}}{4})} |v(s,y)|^2 dy \right)^{p/2} dx \right)^{1/p}$$
$$\lesssim \left(\int_{\mathbb{R}^n} \left(\fint_{s/2}^{s} \fint_{B(x,\frac{\sqrt{s}}{2})} |v(\tau,y)|^2 d\tau dy \right)^{p/2} dx \right)^{1/p} \lesssim s^{\beta+1/2} \|v\|_{T^p_{\beta+1}}. \tag{6.74}$$

Furthermore, using the change of aperture for slice spaces (cf. (3.2)), we get

$$\|v(s)\|_{E^p_\delta} \lesssim \|v(s)\|_{E^p_{s/16}} \lesssim s^{\beta+1/2} \|v\|_{T^p_{\beta+1}}.$$

Therefore,

$$\lim_{s\to 0} v(s) = 0 \quad \text{ in } E^p_\delta.$$

On the other hand, we apply Lemma 6.49 to L^* to get that

$$\lim_{s\to 0} (e^{-(t-s)L})^* h = \lim_{s\to 0} e^{-(t-s)L^*} h = e^{-tL^*} h, \quad \text{ in } E^{p'}_\delta.$$

Using Lemma 6.48 with $X = E^p_\delta$, we obtain that the right-hand side of (6.73) converges to 0 as $s \to 0$, and hence $v(t) = 0$ in $\mathscr{D}'(\mathbb{R}^n)$.

Case 2: $p = \infty$ Let $\delta > 0$ be a constant so that $0 < s < t < \delta$. Note that

$$\|v(s)\|_{E^\infty_\delta} \lesssim \|v(s)\|_{E^\infty_{s/16}} = \operatorname*{esssup}_{x\in\mathbb{R}^n} \left(\fint_{B(x,\frac{\sqrt{s}}{4})} |v(s,y)|^2 dy \right)^{1/2}$$
$$\lesssim \operatorname*{esssup}_{x\in\mathbb{R}^n} \left(\fint_{s/2}^{s} \fint_{B(x,\frac{\sqrt{s}}{2})} |v(\tau,y)|^2 d\tau dy \right)^{1/2} \lesssim s^{\beta+1/2} \|v\|_{T^\infty_{\beta+1}}.$$

One thus gets

$$\lim_{s\to 0} v(s) = 0 \quad \text{ in } E^\infty_\delta.$$

Meanwhile, the same deduction as in Case 1 gives

$$\lim_{s\to 0} (e^{-(t-s)L})^* h = e^{-tL^*} h, \quad \text{ in } E^1_\delta,$$

and hence also $v(t) = 0$ in $\mathscr{D}'(\mathbb{R}^n)$ for any $t > 0$.

Case 3: $p^\flat_L(\beta) < p < 2$ We claim that

$$\|v(t)\|_p \lesssim t^{\beta+1/2}\|v\|_{T^p_{\beta+1}}, \quad \forall 0 < p \leq 2, \ \forall t > 0. \tag{6.75}$$

Indeed, using Hölder's inequality and (6.74), we have

$$\|v(t)\|_p = \left(\int_{\mathbb{R}^n} \fint_{B(x,\frac{\sqrt{t}}{4})} |v(t,y)|^p dydx\right)^{1/p} \lesssim t^{\beta+1/2}\|v\|_{T^p_{\beta+1}}.$$

The claim hence follows. Moreover, Lemma 6.3 and Lemma 5.20 imply

$$\begin{aligned}\|v(t)\|^2_{L^2(B(x,R))} &\lesssim \left(\frac{\Lambda_1^2}{\Lambda_0 R^2} + \frac{1}{t}\right) t^{2(\beta+1)} \int_{t/2}^t \|s^{-(\beta+1)}v(s)\|^2_{L^2(B(x,2R))} ds \\ &\lesssim \left(\frac{\Lambda_1^2}{\Lambda_0 R^2} + \frac{1}{t}\right) t^{2(\beta+1)+n[2,p]}\|v\|^2_{T^p_{\beta+1}}.\end{aligned}$$

By letting $R \to \infty$, we get

$$\|v(t)\|_2 \lesssim t^{\beta+\frac{1}{2}-\frac{n}{2}[p,2]}\|v\|_{T^p_{\beta+1}}, \quad \forall 0 < p \leq 2, \ \forall t > 0. \tag{6.76}$$

As $p^\flat_L(\beta) < p < 2$ implies

$$\frac{1}{p} - \frac{2\beta+1}{n} < \frac{1}{p^\flat_-(L)},$$

there exists $q \in (p,2) \cap (p^\flat_-(L), 2)$ such that

$$\frac{1}{p} - \frac{2\beta+1}{n} < \frac{1}{q}.$$

Interpolating (6.75) and (6.76) yields

$$\|v(t)\|_q \lesssim t^{\beta+\frac{1}{2}-\frac{n}{2}[p,q]}\|v\|_{T^p_{\beta+1}}, \quad \forall q \in (p,2), \ \forall t > 0.$$

Thus, we get

$$\lim_{s\to 0} v(s) = 0 \quad \text{in } L^q(\mathbb{R}^n).$$

Thanks to Proposition 6.9, we also know that

$$\lim_{s\to 0} (e^{-(t-s)L})^* h = \lim_{s\to 0} e^{-(t-s)L^*} h = e^{-tL^*} h, \quad \text{in } L^{q'}(\mathbb{R}^n),$$

since $2 < q' < p_+(L^*) = (p_-(L))'$. Using (6.73) and Lemma 6.48 with $X = L^q(\mathbb{R}^n)$, we conclude that $v(t) = 0$ in $\mathscr{D}'(\mathbb{R}^n)$.

This completes the proof. $\square$

6.5.4 Proof of Theorem 6.42

Proof of Theorem 6.42. Let u be a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = 0$ with $\nabla u \in T^p_{\beta+1/2}$ and null initial data. Thanks to Lemma 6.32, we may assume

$$\boxed{p > \max\{\widetilde{p}_L(\beta), 1\}}$$

in what follows. Let us begin by some important common facts.

First, as $p > \widetilde{p}_L(\beta) \geq \frac{n}{n+2\beta+2}$ and $u(0) = 0$, Proposition 6.38 yields

$$u = \mathcal{R}^{-\Delta}_{1/2}((A - \mathbb{I})\nabla u). \tag{6.77}$$

Hence, Theorem 6.22 for $\mathcal{R}^{-\Delta}_{1/2}$ applies: $u \in T^p_{\beta+1}$ and $u(s)$ tends to 0 as $s \to 0$ in $\mathscr{S}'$ and in various finer topology depending on β and p.

Second, as $\nabla u \in T^p_{\beta+1/2}$, we infer from Lemma 5.22 that the growth of the local L^2-norm of u satisfies the conditions in Proposition 6.47. Therefore, we obtain the homotopy identity: For $0 < s < t < \infty$ and $h \in C^\infty_c(\mathbb{R}^n)$, it holds that

$$\int_{\mathbb{R}^n} u(t,x)\overline{h}(x)dx = \int_{\mathbb{R}^n} u(s,x)\overline{e^{-(t-s)L^*}h}(x)dx. \tag{6.78}$$

Now, fix $t > 0$ and $h \in C^\infty_c(\mathbb{R}^n)$. We wish to let $s \to 0$, yielding $u(t) = 0$ in $\mathscr{D}'(\mathbb{R}^n)$. To take advantage of the different modes of convergence established in Theorem 6.22 (d), we split the argument in two cases.

Case 1: $\beta > -1/2$ As $p > \max\{\widetilde{p}_L(\beta), 1\} \geq p^\flat_L(\beta)$ and $u \in T^p_{\beta+1}$, we can use Theorem 6.44 to conclude $u = 0$.

Case 2: $-1 < \beta \leq -1/2$ We begin with $p \geq 2$ and then $\max\{\widetilde{p}_L(\beta), 1\} < p \leq 2$, which is further split into two sub-cases.

Case 2(a): $2 < p \leq \infty$ Pick $\delta > 0$. We use the slice spaces $E^{-1,p}_\delta$ and $E^{1,p'}_\delta$ introduced in Section 3.4.1. Recall that the $L^2(\mathbb{R}^n)$-inner product realizes $E^{-1,p}_\delta$ as the dual of $E^{1,p'}_\delta$. The formula (6.77) and Theorem 6.22 (d) for $\mathcal{R}^{-\Delta}_{1/2}$ imply

$$\lim_{s\to 0} u(s) = 0 \quad \text{in } E^{-1,p}_\delta.$$

Moreover, we claim that

$$\lim_{s\to 0} e^{-(t-s)L^*}h = e^{-tL^*}h \quad \text{in } E^{1,p'}_\delta. \tag{6.79}$$

Then taking limit of the right-hand integral in (6.78) yields $u(t) = 0$.

To prove (6.79), pick $\lambda \in (0,1)$ and $R > \delta > 0$ so that $\operatorname{supp}(h) \subset B := B(0,R)$ and for any $|x| > R$, $D(x) := \operatorname{dist}(B(x,\delta), \operatorname{supp}(h)) \geq \lambda |x|$. If $|x| \leq R$, we have

$$\|\mathbb{1}_{B(x,\delta)} \nabla e^{-tL^*}h\|_2 \leq \|\nabla e^{-tL^*}h\|_2 \lesssim t^{-1/2}\|h\|_2.$$

If $|x| > R$, the exponential $L^2 - L^2$ off-diagonal estimates of $(t^{1/2}\nabla e^{-tL^*})$ yield there exists $\gamma > 0$ so that

$$\|\mathbb{1}_{B(x,\delta)}\nabla e^{-tL^*}h\|_2 \lesssim t^{-1/2}e^{-\frac{\gamma D(x)^2}{t}}\|h\|_2 \lesssim t^{-1/2}e^{-\frac{\gamma\lambda^2|x|^2}{t}}\|h\|_2.$$

We hence get

$$\sup_{0\le s\le t/2}\|\mathbb{1}_{B(x,\delta)}\nabla e^{-(t-s)L^*}h\|_2 \lesssim t^{-\frac{1}{2}}(\mathbb{1}_B(x) + \mathbb{1}_{B^c}(x)e^{-\frac{\gamma\lambda^2|x|^2}{t}})\|h\|_2.$$

Note that the function on the right-hand side lies in $L^{p'}$, so $(\nabla e^{-(t-s)L^*}h)_{0\le s\le t/2}$ form a uniformly bounded set in $E_\delta^{p'}$. Moreover, Lebesgue's dominated convergence theorem and continuity of (∇e^{-tL^*}) on L^2 imply that $\nabla e^{-(t-s)L^*}h$ converges to $\nabla e^{-tL^*}h$ in $E_\delta^{p'}$ as $s \to 0$. Hence equivalently, $e^{-(t-s)L^*}h$ converges to $e^{-tL^*}h$ in $E_\delta^{1,p'}$ as $s \to 0$. This proves (6.79).

Case 2(b): $p_-(2\beta+1, L) < p \le 2$ Recall that we also assume $p > 1$, and we know that $\widetilde{p}_{-\Delta}(\beta) \le p_-(2\beta+1, L)$, so in particular, we have

$$\max\{\widetilde{p}_{-\Delta}(\beta), 1\} \le \max\{p_-(2\beta+1, L), 1\} < p \le 2.$$

Using the formula (6.77) and Theorem 6.22 (d) for $\mathcal{R}_{1/2}^{-\Delta}$, we get

$$\lim_{s\to 0} u(s) = 0 \quad \text{in } \dot{H}^{2\beta+1,p}.$$

On the other hand, as $h \in \dot{H}^{-(2\beta+1),p'} \cap L^2$ and

$$2 \le p' < (\max\{p_-(2\beta+1, L), 1\})' = p_+(-(2\beta+1), L^*),$$

we infer from Theorem 6.33 (d) that

$$\lim_{s\to 0} e^{-(t-s)L^*}h = \lim_{s\to 0}\mathcal{E}_{L^*}(h)(t-s) = e^{-tL^*}h \quad \text{in } \dot{H}^{-(2\beta+1),p'}.$$

Realizing the right-hand integral of (6.78) in the sense of duality for $\dot{H}^{2\beta+1,p}$ and $\dot{H}^{-(2\beta+1),p'}$, we obtain that it tends to 0 as $s \to 0$.

Case 2(c): $\widetilde{p}_L(\beta) < p \le p_-(2\beta+1, L)$ This case only occurs when $p_-(L) < 1$. Theorem 6.22 (d) still gives us a limit $u(s) \to 0$ in $\dot{H}^{2\beta+1,p}$ but now we do not know whether $e^{-(t-s)L^*}h$ tends to $e^{-tL^*}h$ in the dual space. However, the extra embedding in Proposition 6.36 (ii) allows us to enhance the convergence of $u(s)$.

Lemma 6.50 (Null limit of $u(s)$ in $\mathbb{H}_L^{2\beta+1,p}$). *In the range determined by this case, for all $t > 0$, $u(t)$ lies in $\mathbb{H}_L^{2\beta+1,p}$ with a uniform bound. Moreover,*

$$\lim_{t\to 0} u(t) = 0 \quad \textit{in } \mathbb{H}_L^{2\beta+1,p}.$$

The proof is presented right after. Admitting this lemma, let us show $u = 0$. Since $u(s) \in \mathbb{H}_L^{2\beta+1,p}$ for all $s > 0$, we have $u(s) \in L^2$ and we get from (6.78) that $u(t) = e^{-(t-s)L}u(s)$ for $t \geq s$. By Lemma 6.37, we get

$$\sup_{t\geq s} \|u(t)\|_{\mathbb{H}_L^{2\beta+1,p}} \lesssim \|u(s)\|_{\mathbb{H}_L^{2\beta+1,p}},$$

with a uniform bound in s. As the right-hand side tends to 0 as $s \to 0$, we conclude that $\sup_{t>0} \|u(t)\|_{\mathbb{H}_L^{2\beta+1,p}} = 0$, so $u = 0$. □

Proof of Lemma 6.50. The proof is divided into 4 steps.

Step 1: Regularity of u We prove $u \in C^\infty((0,\infty); L^p \cap L^2)$. Since $1 < p < 2$, as proved in (6.75), for a.e. $s > 0$, we have $u(s) \in L^p$ with

$$\|u(s)\|_p \lesssim s^{\beta+1/2} \|u\|_{T^p_{\beta+1}}.$$

For such an s, L^p-boundedeness of (e^{-tL}) yields $e^{-(t-s)L}u(s) \in L^p$ for all $t \geq s$, and it follows from (6.78) that it equals to $u(t)$. Thus we obtain $u(t) = e^{-(t-s)L}u(s)$ for any $t \geq s > 0$. Applying analyticity of (e^{-tL}) on L^p, we get $u \in C^\infty((0,\infty); L^p)$. As e^{-tL} also maps L^p to L^2 for any $t > 0$, we have $u \in C^\infty((0,\infty); L^2)$ as desired.

Step 2: A key estimate For any $k \geq 1$, we show that

$$\|t^k(\partial_t^k u)(t)\|_{T^p_{\beta+1}} \lesssim_k \|u\|_{T^p_{\beta+1}}. \tag{6.80}$$

We prove it for $k = 1$, and iteration concludes the argument. Observe that by Step 1, we have that for any $t > 0$ and $\tau > 0$,

$$(\partial_t u)(t+\tau) = \partial_\tau(u(t+\tau)) = \partial_\tau(e^{-\tau L}u(t)) = -Le^{-\tau L}u(t). \tag{6.81}$$

In particular, pick $\tau = t$, we get $(\partial_t u)(2t) = -Le^{-tL}u(t)$, and hence,

$$\begin{aligned}\|t\partial_t u\|_{T^p_{\beta+1}} &\eqsim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \fint_{B(x,(2s)^{1/2})} |s^{-(\beta+1)} s(\partial_t u)(2s,y)|^2\, dsdy\right)^{p/2} dx\right)^{1/p} \\ &= \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \fint_{B(x,(2s)^{1/2})} |s^{-(\beta+1)}(sLe^{-sL}u(s))(y)|^2\, dsdy\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}.\end{aligned}$$

Write $B := B(x,(2s)^{1/2})$, $C_0 := 2B$, and $C_j := 2^{j+1}B \setminus 2^j B$ for $j \geq 1$. The $L^2 - L^2$ off-diagonal estimates of (sLe^{-sL}) yield that

$$\|\mathbb{1}_B sLe^{-sL}u(s)\|_2 \leq \sum_{j\geq 0} \|\mathbb{1}_B sLe^{-sL}\mathbb{1}_{C_j}u(s)\|_2 \lesssim \sum_{j\geq 0} e^{-c2^{2j}} \|\mathbb{1}_{C_j}u(s)\|_2,$$

where $c > 0$ is independent of s. Applying this on the above computation, we get

$$\begin{aligned}
\|t\partial_t u\|_{T^p_{\beta+1}} &\lesssim \sum_{j\geq 0} e^{-c2^{2j}} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \int_{C_j} s^{-\frac{n}{2}} |s^{-(\beta+1)} u(s,y)|^2 \, ds dy \right)^{p/2} dx \right)^{1/p} \\
&\lesssim \sum_{j\geq 0} 2^{j\frac{n}{2}} e^{-c2^{2j}} \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \fint_{B(x,2^{j+1}(2s)^{1/2})} |s^{-(\beta+1)} u(s,y)|^2 \, ds dy \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \\
&\lesssim \sum_{j\geq 0} 2^{j(\frac{n}{2}+\frac{n}{p})} e^{-c2^{2j}} \|u\|_{T^p_{\beta+1}} \lesssim \|u\|_{T^p_{\beta+1}}.
\end{aligned}$$

The factor $2^{j\frac{n}{p}}$ in the third inequality comes from change of aperture for tent space norms, see Lemma 3.12.

Step 3: Boundedness of $u(t)$ in $\mathbb{H}^{2\beta+1,p}_L$ Now we prove that for any $t > 0$, $u(t)$ belongs to $\mathbb{H}^{2\beta+1,p}_L$ with

$$\sup_{t>0} \|u(t)\|_{\mathbb{H}^{2\beta+1,p}_L} \lesssim \|u\|_{T^p_{\beta+1}}. \tag{6.82}$$

By Step 1, $u(t)$ lies in L^2 for any $t > 0$, so we just need to show the norm estimate. To this end, pick an integer $k \geq 1$ with $(k-\beta-1)p - \frac{n}{2} > 0$. From [AE23a, §8.2], $\psi(z) := z^k e^{-z}$ is a valid admissible function for $\mathbb{H}^{2\beta+1,p}_L$. So we use (6.80), (6.81) in Step 2, and the identity $\partial^k_\tau\big(u(t+\tau)\big) = (\partial^k_t u)(t+\tau)$ to deduce that for any $t > 0$,

$$\begin{aligned}
&\|u(t)\|_{\mathbb{H}^{2\beta+1,p}_L} \\
&\eqsim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \fint_{B(x,\tau^{1/2})} |\tau^{-(\beta+1)} \left((\tau L)^k e^{-\tau L} u(t) \right)(y)|^2 \, dy d\tau \right)^{p/2} dx \right)^{1/p} \\
&\eqsim \left(\int_{\mathbb{R}^n} \left(\int_0^\infty \fint_{B(x,\tau^{1/2})} |\tau^{-(\beta+1)} \tau^k \partial^k_\tau \big(u(t+\tau,y)\big)|^2 \, dy d\tau \right)^{p/2} dx \right)^{1/p} \\
&\eqsim \left(\int_{\mathbb{R}^n} \left(\int_t^\infty \int_{B(x,(\sigma-t)^{1/2})} (\sigma-t)^{(k-\beta-1)p-\frac{n}{2}} |(\partial^k_t u)(\sigma,y)|^2 \, dy d\sigma \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \\
&\lesssim \|\sigma^k (\partial^k_t u)(\sigma)\|_{T^p_{\beta+1}} \lesssim \|u\|_{T^p_{\beta+1}}.
\end{aligned}$$

The first inequality with implicit constant independent of t uses $(k-\beta-1)p - \frac{n}{2} > 0$. This proves (6.82).

Step 4: Limit at $t = 0$ We finish by showing that $u(t)$ tends to 0 in $\mathbb{H}_L^{2\beta+1,p}$ as $t \to 0$. Suppose $(k-\beta-1)p - \frac{n}{2} > 0$. Remark that examination of the argument in Step 2 allows time truncation. More precisely, if $0 < t \le \delta$, then one finds

$$\|\mathbb{1}_{\{\tau<2\delta\}}\, \partial_\tau^k(u(t+\tau))\|_{T^p_{\beta+1-k}} \lesssim \|\mathbb{1}_{\{\tau<\delta\}}\, u\|_{T^p_{\beta+1}},$$

where the implicit constant does not depend on t, δ. Using Newton–Leibniz formula for L^2-valued functions, we get

$$\tau^k\partial_\tau^k(u(t+\tau)) - \tau^k\partial_\tau^k(u(t'+\tau)) = \frac{1}{\tau}\int_{t'}^{t} \tau^{k+1}\partial_\tau^{k+1}(u(h+\tau))\, dh.$$

Also, following the argument of Step 2 with the help of Minkowski's integral inequality, we find for $t, t' \in (0, \delta]$,

$$\|\mathbb{1}_{\{\tau<2\delta\}}\, (\partial_\tau^k(u(t+\tau)) - \partial_\tau^k(u(t'+\tau)))\|_{T^p_{\beta+1-k}} \lesssim \frac{|t-t'|}{\delta}\|u\|_{T^p_{\beta+1}},$$

again with implicit constant independent of t, t', δ. This implies that $((\tau, x) \mapsto \partial_\tau^k(u(t+\tau, x)))_{t>0}$ is Cauchy in $T^p_{\beta+1-k}$ when $t \to 0$. We deduce as in Step 3 that $(u(t))_{t>0}$ is Cauchy in $\mathbb{H}_L^{2\beta+1,p}$ when $t \to 0$. As mentioned in the first paragraph of Case 2(c), $u(t)$ tends to 0 in $\dot{H}^{2\beta+1,p}$. The conditions on β, p allow us to apply Proposition 6.36 (ii), so we conclude that the limit of $u(t)$ exists and must be zero in $\mathbb{H}_L^{2\beta+1,p}$. This completes the proof. □

6.5.5 Representation

To finish this section, we provide the proof of Theorem 6.43.

Proof of Theorem 6.43. Let u be a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = 0$ with $\nabla u \in T^p_{\beta+1/2}$. As $p > \widetilde{p}_L(\beta) \ge \frac{n}{n+2\beta+2}$, Proposition 6.38 asserts that there exists $u_0 \in \mathscr{S}'$ so that

$$u = \mathcal{E}_{-\Delta}(u_0) + \mathcal{R}_{1/2}^{-\Delta}((A-\mathbb{I})\nabla u).$$

In case (i), as $\beta \ge 0$ and $\frac{n}{n+2\beta} \le p \le \infty$, Proposition 6.38 (i) says u_0 is a constant. Hence, $w := u - \mathcal{E}_{-\Delta}(u_0)$ is a global weak solution to the Cauchy problem

$$\begin{cases} \partial_t w - \operatorname{div}(A\nabla w) = 0, \\ w(0) = 0 \end{cases}$$

with $\nabla w \in T^p_{\beta+1/2}$. As $\widetilde{p}_L(\beta) < p \le \infty$, we invoke Theorem 6.42 to get $w = 0$, so $u = \mathcal{E}_{-\Delta}(u_0)$ is a constant as desired.

In case (ii), Proposition 6.38 (ii) shows that there exist $g \in \dot{H}^{2\beta+1,p}$ and $c \in \mathbb{C}$ so that $u_0 = g + c$. Then we apply Theorem 6.33 to g to get

$$u = \mathcal{E}_{-\Delta}(g) + \mathcal{R}_{1/2}^{-\Delta}((A-\mathbb{I})\nabla u) + c = \mathcal{E}_L(g) + c.$$

This completes the proof. □

6.6 Results for homogeneous Besov spaces

As seen for the heat equation in Section 5.5, our results for more general parabolic equations of type (6.1) also extend to the context of homogeneous Besov spaces and Z-spaces.

Recall that at the exceptional point $\beta = 0$ (*i.e.*, $s = 1$) and $p = \infty$, Proposition 5.24 yields the homogeneous Cauchy problem for the heat equation is still well-posed for initial data in $\dot{W}^{1,\infty}$ (rather than $\dot{B}^1_{\infty,\infty}$). We call it as the *canonical modification* for $\beta = 0$ and $p = \infty$.

Theorem 6.51 (Well-posedness of Cauchy problems of type (6.1) for homogeneous Besov spaces)**.** *Let $\beta > -1$ and $0 < p \leq \infty$. With canonical modifications for $\beta = 0$ and $p = \infty$ (i.e., the initial data u_0 lies in $\dot{W}^{1,\infty}$), Theorems 6.11, 6.22, 6.33, 6.42, and 6.43, and 6.44 are all valid in the same range of exponents, when replacing homogeneous Hardy–Sobolev spaces $\dot{H}^{2\beta+1,p}$ (resp. tent spaces $T^p_{\beta+1/2}$) by homogeneous Besov spaces $\dot{B}^{2\beta+1}_{p,p}$ (resp. Z-spaces $Z^p_{\beta+1/2}$) with the same exponents.*

Proof. We just provide some key ingredients of the proof. For convenience, we still use the same label of the theorems for their variants in Besov spaces.

First consider Theorem 6.11. Let $\beta > -1/2$ and $p^\flat_L(\beta) < p \leq \infty$. Pick β_0 and β_1 so that

$$\beta_0 < \beta < \beta_1, \quad p > p^\flat_L(\beta_0) > p^\flat_L(\beta_1).$$

Let $\kappa = 1, 1/2, 0$. Since $\mathcal{L}^L_\kappa$ is bounded from $T^p_{\beta_i}$ to $T^p_{\beta_i+\kappa}$ for $i = 0, 1$, by real interpolation, one gets $\mathcal{L}^L_\kappa$ is bounded from Z^p_β to $Z^p_{\beta+\kappa}$. This gives all the needed estimates to establish (a), (b), (c), and (e). It only remains to prove (d). For $p < \infty$, it follows from the Sobolev embedding of tent spaces, see Proposition 3.10. For $p = \infty$, it follows from direct computation. Let $u \in Z^\infty_{\beta+1}$. For a.e. $x \in \mathbb{R}^n$, we have

$$\fint_{W(t,x)} |u(s,y)|^2 dsdy \lesssim t^{2\beta+1} \|u\|^2_{Z^\infty_{\beta+1}},$$

which tends to 0 as $t \to 0$, since $\beta > -1/2$. This proves Theorem 6.11.

Theorem 6.22 can also be proved similarly, but only one exceptional case needs extra work: For $\beta = -1/2$, $\widetilde{p}_L(-1/2) = p_-(L) < p \leq 2$, and $F \in Z^p_0$, one needs to show $u = \mathcal{R}^L_{1/2}(F)$ tends to 0 as $t \to 0$ in L^p. Let us verify this. We have shown in (6.54) that

$$\|u(t)\|^p_p \lesssim \int_{\mathbb{R}^n} \left(\fint_{t/2}^t \fint_{B(x,\frac{\sqrt{t}}{2})} |u|^2 \right)^{p/2} dx + \int_{\mathbb{R}^n} \left(\int_{t/2}^t \fint_{B(x,\frac{\sqrt{t}}{2})} |F|^2 \right)^{p/2} dx.$$

Denote by $\Phi_1(t)$ and $\Phi_2(t)$ for the first and the second term, respectively. Notice that

$$\int_0^\infty \Phi_1(t) \frac{dt}{t} \eqsim \|u\|^p_{Z^p_{1/2}}, \quad \int_0^\infty \Phi_2(t) \frac{dt}{t} \eqsim \|F\|^p_{Z^p_0}.$$

The fact that both of the integrals converge forces that both $\Phi_1(t)$ and $\Phi_2(t)$ tend to 0 as $t \to 0$. We hence get $u(t)$ tends to 0 in L^p as desired.

To prove Theorem 6.33, we also use real interpolation. There are only two points that need extra explanations. First, for the exceptional point $\beta = -1/2$ and $p = \infty$, one uses Proposition 5.24. Second, to prove (d), one can adopt L-adapted Besov spaces $\mathbb{B}^s_{p,p}$, see [AE23a, §8.2 and 19.1]. By real interpolation, they share the same identification range and extra embedding range with $\mathbb{H}^{s,p}$, which are explicitly shown in Proposition 6.36. Hence, one can correspondingly establish Lemma 6.37 for $\mathbb{B}^s_{p,p}$ and $\dot{B}^s_{p,p}$ by verbatim adaptation. Then (d) again follows from uniqueness of extension by density.

Next, consider Theorem 6.42. For $p \neq \infty$, it follows from Sobolev embedding of tent spaces, see Proposition 3.10 (iii). For $p = \infty$, one needs to repeat the proof manually. Following the proof of Theorem 6.42, we may suppose $u = \mathcal{R}^{-\Delta}_{1/2}((A - \mathbb{I})\nabla u)$ (cf. (6.77)). We hence infer from Theorem 6.22 (d) that $u(s)$ tends to 0 as $s \to 0$ in $E^{-1,\infty}_\delta$. Thus, using the homotopy identity (6.78) and the strong continuity of the semigroup (e^{-tL^*}) in $E^{1,1}_\delta$ (cf. (6.79)), we conclude that $u = 0$.

Theorem 6.44 follows from the same argument as for Theorem 6.42, using the Sobolev embedding of tent spaces for $p < \infty$ and the trace in Theorem 6.11 manually for $p = \infty$.

Finally, Theorem 6.43 follows by combining Theorem 5.25 with the arguments in Section 6.5.5.

This completes the proof. □

6.7 An endpoint case $\beta = -1$

This section is devoted to the existence of global weak solutions to the homogeneous Cauchy problem for $\beta = -1$.

Proposition 6.52. *Let $p_-(-1, L) = q_+(L^*)' < p \leq \infty$. For any $v_0 \in \dot{H}^{-1,p}$, there is a global weak solution $v \in C([0,\infty); \mathscr{S}')$ to (HC) with initial data v_0 so that*

$$\|\nabla v\|_{T^p_{-1/2}} \lesssim \|v\|_{T^p_0} \lesssim \|v_0\|_{\dot{H}^{-1,p}}. \tag{6.83}$$

Moreover, if $p_-(-1, L) < p < p_+(-1, L)$, then $v \in C([0,\infty); \dot{H}^{-1,p})$.

Remark 6.53. For $\beta = -1$, neither $\mathcal{R}^L_{1/2}$ nor $\mathcal{R}^{-\Delta}_{1/2}$ is defined on $T^p_{-1/2}$, so Lemma 6.39 fails, and we do not have methods to prove uniqueness or representation for general parabolic equations. However, it is possible for the heat equation; for example, a representation result is presented in Theorem 5.16 for $v \in T^\infty_0$ with trace $v_0 \in \dot{H}^{-1,\infty} \simeq \mathrm{BMO}^{-1}$. Similarly, we do not know the converse inequality in (6.83).

Our main strategy here is to define the operator $\mathcal{E}_L$ on $\dot{H}^{-1,p}$ by an extension of $(e^{-tL}\operatorname{div})_{t>0}$ acting on $\dot{H}^{0,p}$, since any $\dot{H}^{-1,p}$-distributions can be written as the divergence of $\dot{H}^{0,p}$-functions. This fact follows from boundedness of Riesz transforms on L^p when $1<p<\infty$ and on BMO when $p=\infty$. Here and in the sequel, we omit to specify that the operator applies to $\mathbb{C}^n$-valued functions.

Lemma 6.54. *Let $q_+(L^*)' < p \le \infty$ and $t>0$.*

(i) *The operator $e^{-tL}\operatorname{div} : W^{1,2} \to W^{1,2}$ extends to a bounded operator from $\dot{H}^{0,p}$ to $W^{1,2}_{\mathsf{loc}}$, denoted by $\mathcal{G}_t$.*

(ii) *For any $f \in \dot{H}^{0,p}$, $\mathcal{G}_t(f)$ converges to $\operatorname{div} f$ in $\mathscr{S}'$ as $t \to 0$.*

Proof. The extension in (i) is constructed by duality. Let B be a ball in $\mathbb{R}^n$ and $\phi \in L^2(B)$. When $q_+(L^*)' < p \le 2$, $L^2 - L^{p'}$ estimates of $(t^{1/2}\nabla e^{-tL^*})$ yields

$$\|\nabla e^{-tL^*}\phi\|_{p'} \lesssim t^{-\frac{1}{2}-\frac{n}{2}[p',2]}\|\phi\|_2.$$

When $p=\infty$, standard molecular estimates combining $L^2 - L^2$ off-diagonal estimates of $(t^{1/2}\nabla e^{-tL^*})$ and the fact that $\nabla e^{-tL^*}\phi$ has mean value 0 imply

$$\|\nabla e^{-tL^*}\phi\|_{H^1} \lesssim (t^{-1/2}|B|^{1/2} + t^{\frac{n}{4}-\frac{1}{2}})\|\phi\|_2.$$

By interpolation, we have $\|\nabla e^{-tL^*}\phi\|_{\dot{H}^{0,p'}} \lesssim_{t,B} \|\phi\|_2$ for $q_+(L^*)' < p \le \infty$. Thus, for any $t>0$, we define $\mathcal{G}_t : \dot{H}^{0,p} \to L^2_{\mathsf{loc}}$ by the pairing

$$\langle \mathcal{G}_t(f), \phi\rangle := \langle f, \nabla e^{-tL^*}\phi\rangle, \quad \forall \phi \in L^2_{\mathsf{c}}(\mathbb{R}^n).$$

Meanwhile, note that $(t\nabla e^{-tL^*}\operatorname{div})$ also has $L^2 - L^{p'}$ off-diagonal estimates for $q_+(L^*)' < p \le 2$, considering the decomposition

$$t\nabla e^{-tL^*}\operatorname{div} = (t^{1/2}\nabla e^{-\frac{t}{2}L^*})(t^{1/2}e^{-\frac{t}{2}L^*}\operatorname{div}). \tag{6.84}$$

So repeating the above argument for $(t\nabla e^{-tL^*}\operatorname{div})$ yields $\nabla\mathcal{G}_t(f)$ also lies in L^2_{loc}, hence $\mathcal{G}_t(f) \in W^{1,2}_{\mathsf{loc}}$. This proves (i).

Next, we proceed with (ii). Pick $\phi \in \mathscr{S}$. It suffices to prove $\nabla e^{-tL^*}\phi$ tends to $\nabla\phi$ in $\dot{H}^{0,p'}$ when $t \to 0$. For $q_+(L^*)' < p \le 2$, as $2 \le p' < q_+(L^*) = p_+(1,L^*)$, it follows by continuity of the semigroup (e^{-tL^*}) on $H^{1,p'}$, see Lemma 6.37. For $p=\infty$, we first assert the following

Lemma 6.55. *Let $\psi \in C^\infty_{\mathsf{c}}(\mathbb{R}^n)$ and B be a ball in $\mathbb{R}^n$ with $\operatorname{supp}(\psi) \subset B$. Then*

$$\|\nabla e^{-tL^*}\psi - \nabla\psi\|_{H^1} \lesssim |B|^{1/2}\|\nabla e^{-tL^*}\psi - \nabla\psi\|_2 + t^{n/4}\|\nabla\psi\|_2.$$

The proof is provided right below. Admitting it, let us show that for any $\phi \in \mathscr{S}$, $\nabla e^{-tL^*}\phi$ tends to $\nabla\phi$ in H^1 as $t \to 0$. Indeed, write $B := B(0,1)$, $C_0 := 4B$, and $C_j := 2^{j+2}B \setminus 2^{j-2}B$ for $j \geq 1$. Let (χ_j) be a smooth partition of unity so that $\mathrm{supp}(\chi_j) \subset C_j$ with $\|\chi_j\|_\infty \leq 1$ and $\|\nabla\chi_j\|_\infty \lesssim 1$. Define $\phi_j := \phi\chi_j$. Note that

$$\begin{aligned}\|\nabla\phi_j\|_2 &\lesssim \left(\int_{C_j} \langle y\rangle^{-2N} |\langle y\rangle^N \nabla\phi(y)|^2 dy\right)^{1/2} + \left(\int_{C_j} \langle y\rangle^{-2N} |\langle y\rangle^N \phi(y)|^2 dy\right)^{1/2} \\ &\lesssim 2^{-j(N-\frac{n}{2})}\mathcal{P}_{N+1}(\phi),\end{aligned} \tag{6.85}$$

where $\mathcal{P}_{N+1}$ is the semi-norm on $\mathscr{S}$ defined in (5.8). Then applying Lemma 6.55 on $\psi = \phi_j$, we get

$$\begin{aligned}\|\nabla e^{-tL^*}\phi - \nabla\phi\|_{H^1} &\lesssim \sum_{j\geq 0} \|\nabla e^{-tL^*}\phi_j - \nabla\phi_j\|_{H^1} \\ &\lesssim \sum_{j\geq 0} 2^{jn/2}\|\nabla e^{-tL^*}\phi_j - \nabla\phi_j\|_2 + t^{n/4}\|\nabla\phi_j\|_2.\end{aligned} \tag{6.86}$$

Using boundedness of (e^{-tL^*}) on $\dot{H}^{1,2}$ and (6.85), for $t \leq 1$, we have

$$\sum_{j\geq 0} 2^{jn/2}\|\nabla e^{-tL^*}\phi_j - \nabla\phi_j\|_2 + t^{n/4}\|\nabla\phi_j\|_2 \lesssim \sum_{j\geq 0} 2^{-j(N-n)}\mathcal{P}_{N+1}(\phi),$$

which converges when $N > n$. Therefore, Lebesgue's dominated convergence theorem implies the right-hand side in (6.86) tends to 0 as $t \to 0$, noting that the first term tends to 0 by continuity of (e^{-tL^*}) on $\dot{H}^{1,2}$. This completes the proof. □

Proof of Lemma 6.55. Write $C_0 := 2B$ and $C_j := 2^{j+1}B \setminus 2^jB$ for $j \geq 1$. Consider the molecular decomposition

$$\begin{aligned}\nabla e^{-tL^*}\psi - \nabla\psi = \mathbb{1}_{C_0}(\nabla e^{-tL^*}\psi - \nabla\psi) &+ \sum_{j\geq 1} \mathbb{1}_{C_j}\nabla e^{-tL^*}\mathbb{1}_B(\psi - \langle\psi\rangle) \\ &+ \sum_{j\geq 1} \mathbb{1}_{C_j}\nabla e^{-tL^*}(\mathbb{1}_B\langle\psi\rangle),\end{aligned}$$

where $\langle\psi\rangle := \fint_B \psi$. For $j \geq 1$, using $L^2 - L^2$ off-diagonal estimates of $(t^{1/2}\nabla e^{-tL^*})$ and Poincaré's inequality, one gets

$$\begin{aligned}\left\|\mathbb{1}_{C_j}\nabla e^{-tL^*}\mathbb{1}_B(\psi - \langle\psi\rangle)\right\|_2 &\lesssim 2^{-j}t^{\frac{n}{4}}\left(\frac{2^jr(B)}{t^{1/2}}\right)^{\frac{n}{2}+1} e^{-\frac{c(2^jr(B))^2}{t}}\|\nabla\psi\|_2|2^{j+1}B|^{-\frac{1}{2}} \\ &\lesssim 2^{-j}t^{n/4}\|\nabla\psi\|_2|2^{j+1}B|^{-1/2}.\end{aligned}$$

Moreover, since $\psi \in C_c^\infty(\mathbb{R}^n)$ with $\mathrm{supp}(\psi) \subset B$, one has

$$\fint_B |\psi| \lesssim r(B)^{-n+1}\int_B |\nabla\psi| \lesssim r(B)^{-n/2+1}\|\nabla\psi\|_2.$$

Then, using $L^2 - L^2$ off-diagonal estimates of $(t^{1/2}\nabla e^{-tL^*})$ again, we obtain

$$\begin{aligned}\left\|\mathbb{1}_{C_j}\nabla e^{-tL^*}(\mathbb{1}_B\langle\psi\rangle)\right\|_2 &\lesssim 2^{-j}t^{\frac{n}{4}}\left(\frac{2^j r(B)}{t^{1/2}}\right)^{\frac{n}{2}+1} e^{-\frac{c(2^j r(B))^2}{t}}\|\nabla\psi\|_2|2^{j+1}B|^{-\frac{1}{2}} \\ &\lesssim 2^{-j}t^{n/4}\|\nabla\psi\|_2|2^{j+1}B|^{-1/2}.\end{aligned}$$

Therefore, gathering all the estimates gives the inequality desired. □

Let us finish by proving Proposition 6.52.

Proof of Proposition 6.52. Let $q_+(L^*)' < p \leq \infty$. Define the operator $\mathcal{G} : \dot{H}^{0,p} \to L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$ by

$$\mathcal{G}(f)(t) := \mathcal{G}_t(f). \tag{6.87}$$

We first show that $v := \mathcal{G}(f)$ satisfies

$$\|v\|_{T^p_0} \lesssim \|f\|_{\dot{H}^{0,p}}. \tag{6.88}$$

When $q_+(L^*)' < p \leq 2$, we know from [AE23a, §8.2] that $\psi(z) := z^{1/2}e^{-z}$ is an admissible function (see Section 6.4.2) for $s \leq 0$ and $p \leq 2$. We hence infer from L^p-boundedness of $L^{-1/2}\operatorname{div}$ (see *e.g.*, [Aus07, Theorem 4.1]) that

$$\|\mathcal{G}(f)\|_{T^p_0} = \|(tL)^{1/2}e^{-tL}L^{-1/2}\operatorname{div} f\|_{T^p_{1/2}} \eqsim \|L^{-1/2}\operatorname{div} f\|_p \lesssim \|f\|_p. \tag{6.89}$$

Next, when $p = \infty$, we use a classical argument on the relation between Carleson measures and BMO-functions. Let f be in $\dot{H}^{0,\infty} \simeq \mathrm{BMO}$. Recall that

$$\|\mathcal{G}(f)\|_{T^\infty_0} = \sup_B\left(\frac{1}{|B|}\int_0^{r(B)^2}\int_B|\mathcal{G}_t(f)(y)|^2 dtdy\right)^{1/2}.$$

Let B be a ball in $\mathbb{R}^n$. Write $C_0 := 2B$ and $C_j := 2^{j+1}B \setminus 2^jB$ for any $j \geq 1$. Consider the decomposition

$$f = \langle f\rangle_{C_0} + \sum_{j\geq 0}(f - \langle f\rangle_{C_0})\mathbb{1}_{C_j} =: \langle f\rangle_{C_0} + \sum_{j\geq 0} f_j \quad \text{in } \mathscr{D}',$$

where $\langle f\rangle_{C_0} := ⨍_{C_0} f$. As $f_j \in L^2$, by definition, one finds that for any $t > 0$, $\mathcal{G}(f)(t) = \sum_{j\geq 0}\mathcal{G}(f_j)(t) = \sum_{j\geq 0} e^{-tL}\operatorname{div} f_j$ in $\mathscr{D}'$, so we get

$$\|\mathcal{G}(f)\|_{L^2((0,r(B)^2)\times B)} \lesssim \sum_{j\geq 0}\|e^{-tL}\operatorname{div} f_j\|_{L^2((0,r(B)^2)\times B)} =: I_j.$$

For I_0, this follows from (6.89) as

$$I_0 \leq \|t^{1/2}e^{-tL}\operatorname{div} f_0\|_{L^2_{1/2}(\mathbb{R}^{1+n}_+)} \lesssim \|f_0\|_2 = \|f - \langle f\rangle_{C_0}\|_{L^2(C_0)} \lesssim |B|^{1/2}\|f\|_{\mathrm{BMO}}.$$

For $j \geq 1$, we infer from $L^2 - L^2$ off-diagonal estimates of $(t^{1/2}e^{-tL}\operatorname{div})$ that

$$
\begin{aligned}
I_j &\lesssim \left(\int_0^{r(B)^2} e^{-2c\frac{(2^j r(B))^2}{t}} \frac{dt}{t} \int_{C_j} |f - \langle f \rangle_{C_0}|^2 \right)^{1/2} \\
&\lesssim \left(\int_0^{r(B)^2} |2^{j+1}B| e^{-2c\frac{(2^j r(B))^2}{t}} \frac{dt}{t} \fint_{2^{j+1}B} |f - \langle f \rangle_{2^{j+1}B}|^2 \right)^{1/2} \\
&\quad + \left(\int_0^{r(B)^2} |2^{j+1}B| e^{-2c\frac{(2^j r(B))^2}{t}} \frac{dt}{t} \right)^{1/2} |\langle f \rangle_{2^{j+1}B} - \langle f \rangle_{2B}| \\
&\lesssim 2^{2jn} \log(1+j) e^{-c2^{2j}} |B|^{1/2} \|f\|_{\mathrm{BMO}}.
\end{aligned}
$$

Gathering the estimates, we obtain $\|\mathcal{G}(f)\|_{L^2((0,r(B)^2)\times B)} \lesssim |B|^{1/2}\|f\|_{\mathrm{BMO}}$. Note that the controlling constant is independent of the ball B, so by taking supremum over all balls B in $\mathbb{R}^n$, we have $\|\mathcal{G}(f)\|_{T_0^\infty} \lesssim \|f\|_{\mathrm{BMO}}$ as desired. The rest for $2 \leq p < \infty$ hence follows from interpolation. This proves (6.88).

Meanwhile, observe that when $f \in \mathscr{S}_\infty$, v is clearly a global weak solution to $\partial_t v - \operatorname{div} A\nabla v = 0$ on $\mathbb{R}^{1+n}_+$, so by Corollary 6.4, we also have $\|\nabla v\|_{T^p_{-1/2}} \lesssim \|v\|_{T^p_0}$.

Now, for any $v_0 \in \dot{H}^{-1,p}$, pick $V_0 \in \dot{H}^{0,p}$ so that $v_0 = \operatorname{div} V_0$ with $\|v_0\|_{\dot{H}^{-1,p}} \eqsim \|V_0\|_{\dot{H}^{0,p}}$. Using the above bounds for $f = V_0$, we obtain existence of the weak solution and the estimates (6.83) by a standard density argument.

To prove continuity, recall that Lemma 6.54 (ii) says $v(t)$ converges to v_0 in $\mathscr{S}'$ as $t \to 0$. In fact, a similar argument yields $v \in C((0,\infty);\mathscr{S}')$. When $p_-(-1,L) < p < p_+(-1,L)$, the desired stronger continuity follows from Lemma 6.37, due to uniqueness of extensions by density. This completes the proof. □

6.8 Further generalizations

Our results can be generalized in several directions.

For instance, local well-posedness also holds, since the proofs show that one can work on any given strip $[0,T] \times \mathbb{R}^n$ instead.

Let us also briefly mention two possible results. For convenience, we restrict ourselves to inhomogeneous Cauchy problems

$$\partial_t u - \operatorname{div}(A\nabla u) = f.$$

The same deduction may also work for Lions' equation and homogeneous Cauchy problems.

6.8.1 Tent spaces of Carleson measure type

Observe that Corollary 4.14 and Lemma 6.13 imply $\mathcal{L}_1$ extends to a bounded operator from $T^\infty_{\beta,([p,1])}$ to $T^\infty_{\beta+1,([p,1])}$ for any $0 < p \leq 1$. In fact, we can also establish well-posedness within this range using the same method as what we did for $f \in T^\infty_\beta$.

Proposition 6.56. *Let $\beta > -1/2$ and $0 < p \leq 1$. Then for any $f \in T^\infty_{\beta;([p,1])}$, there exists a unique global weak solution $u \in T^\infty_{\beta+1;([p,1])}$ to the equation $\partial_t u - \operatorname{div}(A\nabla u) = f$.*

Similarly, all the properties of Theorem 6.11 can be accordingly established. We only mention that (e) turns out to be that:

(e') $u \in C([0,\infty);\mathscr{S}')$ and $u(t)$ converges to 0 as $t \to 0$ in E^∞_δ for any $\delta > 0$.

6.8.2 Local Kenig–Pipher spaces

Let $\beta \in \mathbb{R}$, $0 < T \leq \infty$, and $u \in L^2_{\mathsf{loc}}((0,T) \times \mathbb{R}^n)$. The *(local weighted) Kenig–Pipher non-tangential maximal function* $\mathcal{N}_{\beta,T}$ is defined by

$$\mathcal{N}_{\beta,T}(u)(x) := \sup_{0<t<T} \left(⨍_{t/2}^{t} ⨍_{B(x,t^{1/2})} |s^{-\beta} u(s,y)|^2 ds dy \right)^{1/2}.$$

For $0 < p \leq \infty$, the *(local weighted) Kenig–Pipher space* $X^p_{\beta,T}$ consists of measurable functions $u \in L^2_{\mathsf{loc}}((0,T) \times \mathbb{R}^n)$ for which

$$\|u\|_{X^p_{\beta,T}} := \|\mathcal{N}_{\beta,T}(u)\|_p < \infty.$$

Simple calculation shows that for $0 < p \leq \infty$, $\beta \in \mathbb{R}$, and $0 < T \leq \infty$,

$$T^p_{\beta+1/2} \hookrightarrow X^p_{\beta,\infty} \hookrightarrow X^p_{\beta,T}.$$

Thus, when $\beta > -1/2$ and $p^\flat_L(\beta) < p \leq \infty$, for any $f \in T^p_\beta$, the global weak solution u given by Theorem 6.11 lies in $X^p_{\beta+1/2,T}$. In fact, only knowing that $\mathbb{1}_{(0,T]} f \in T^p_\beta$ suffices to conclude that $u \in X^p_{\beta+1/2,T}$. Also, there is uniqueness in $X^p_{\beta+1/2,T}$.

Proposition 6.57. *Let $\beta > -1/2$, $T \in (0,\infty]$, and $p^\flat_L(\beta) < p \leq \infty$. Then any weak solution $u \in X^p_{\beta+1/2,T}$ to $\partial_t u - \operatorname{div}(A\nabla u) = 0$ vanishes.*

It suffices to follow our argument of Theorem 6.44, with little modifications. The only noticeable one is that we use (twice) Lemma 5.20 on strips $(\frac{t}{2}, t) \times \mathbb{R}^n$ for $0 < t < T$ with the remark that

$$\|\mathbb{1}_{(\frac{t}{2},t)} u\|_{T^p_{\beta+1}} \eqsim \|\mathbb{1}_{(\frac{t}{2},t)} u\|_{X^p_{\beta+1/2,T}}.$$

Chapter 7

Non-autonomous parabolic Cauchy problems

> "*You know my methods. Apply them, and it will be instructive to compare results.*"
>
> Sherlock Holmes, *The Sign of the Four*, Arthur Conan Doyle

This chapter is devoted to the existence, uniqueness, and representation of weak solutions to the non-autonomous parabolic Cauchy problems

$$\begin{cases} \partial_t u - \operatorname{div}_x(A(t,x)\nabla_x u) = f + \operatorname{div} F, & (t,x) \in (0,\infty) \times \mathbb{R}^n \\ u(0) = u_0 \end{cases}.$$

Assumption 7.1. Throughout this chapter, assume that $A = A(t,x) \in L^\infty((0,\infty) \times \mathbb{R}^n; \operatorname{Mat}_n(\mathbb{C}))$ is *uniformly elliptic*, that is, there exist $\Lambda_0, \Lambda_1 > 0$ so that for a.e. $(t,x) \in (0,\infty) \times \mathbb{R}^n$ and any $\xi, \eta \in \mathbb{C}^n$,

$$\Re(\langle A(t,x)\xi, \xi \rangle) \geq \Lambda_0 |\xi|^2, \quad |\langle A(t,x)\xi, \eta \rangle| \leq \Lambda_1 |\xi||\eta|.$$

This chapter contains the article [Hou25] "*On well-posedness for non-autonomous parabolic Cauchy problems with rough initial data*".

7.1 Basic facts of weak solutions

We first recall the definition of weak solutions. Denote by $W^{1,2}$ the *(inhomogeneous) Sobolev space* endowed with the norm $\|f\|_{W^{1,2}} := \|f\|_{L^2} + \|\nabla f\|_{L^2}$.

Definition 7.2 (Weak solutions)**.** Let Ω be an open subset of $\mathbb{R}^n$ and $0 \le a < b \le \infty$. Let f and F be in $\mathscr{D}'((a,b)\times\Omega)$. A function $u \in L^2_{\mathsf{loc}}((a,b); W^{1,2}_{\mathsf{loc}}(\Omega))$ is called a *weak solution* to the equation

$$\partial_t u - \operatorname{div}(A\nabla u) = f + \operatorname{div} F$$

with *source term* $f + \operatorname{div} F$, if for any $\phi \in C^\infty_{\mathsf{c}}((a,b)\times\Omega)$,

$$-\int_{(a,b)\times\Omega} u\, \partial_t\phi + \int_{(a,b)\times\Omega} (A\nabla u)\cdot\nabla\phi = (f,\phi) - (F,\nabla\phi). \tag{7.1}$$

The pairs on the right-hand side stand for pairing of distributions and test functions on $(a,b)\times\Omega$. We say u is a *global* weak solution if (6.3) holds for $(a,b)\times\Omega = (0,\infty)\times\mathbb{R}^n$.

Let $u_0 \in \mathscr{D}'(\Omega)$. We say u satisfies the *initial condition* $u(a) = u_0$ if $u(t)$ converges to u_0 in $\mathscr{D}'(\Omega)$ as $t \to a+$.

There is a corresponding definition of (global) weak solutions to the backward equation $-\partial_s u - \operatorname{div}(A(s)^*\nabla u) = f + \operatorname{div} F$. We leave the precise formulation to the reader.

7.1.1 Energy inequality

We recall without proof a form of Caccioppoli's inequality.

Lemma 7.3 (Caccioppoli's inequality)**.** *Let $0 < a < b < \infty$ and $B \subset \mathbb{R}^n$ be a ball. Let f and F be in $L^2((a,b)\times 2B)$. Let $u \in L^2((a,b); W^{1,2}(2B))$ be a weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = f + \operatorname{div} F$ in $(a,b)\times 2B$. Then u belongs to $C([a,b]; L^2(B))$ with*

$$\begin{aligned}\|u(b)\|^2_{L^2(B)} \lesssim \left(\frac{1}{r(B)^2} + \frac{1}{b-a}\right)\int_a^b \|u(s)\|^2_{L^2(2B)}\, ds \\ + r(B)^2 \int_a^b \|f(s)\|^2_{L^2(2B)}\, ds + \int_a^b \|F(s)\|^2_{L^2(2B)}\, ds.\end{aligned}$$

Moreover, for any $c \in (a,b)$, it holds that

$$\begin{aligned}\int_c^b \|\nabla u(s)\|^2_{L^2(B)}\, ds \lesssim & \frac{1}{c-a}\left(1 + \frac{b-a}{r(B)^2}\right)\int_a^b \|u(s)\|^2_{L^2(2B)}\, ds \\ & + \frac{r(B)^2(b-a)}{c-a}\int_a^b \|f(s)\|^2_{L^2(2B)}\, ds \\ & + \frac{b-a}{c-a}\int_a^b \|F(s)\|^2_{L^2(2B)}\, ds.\end{aligned}$$

There is a corresponding version for weak solutions to the backward equation $-\partial_s u - \operatorname{div}(A(s)^*\nabla u) = f + \operatorname{div} F$. We refer to "Caccioppoli's inequality" in both cases in the sequel.

Corollary 7.4 (*A priori* tent space estimates)**.** *Let $\beta \in \mathbb{R}$ and $0 < p \leq \infty$. Let u be a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = f + \operatorname{div} F$. Then the following a priori energy inequality holds:*

$$\|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|u\|_{T^p_{\beta+1}} + \|F\|_{T^p_{\beta+1/2}} + \|f\|_{T^p_\beta}.$$

The inequality also holds for weak solutions to the backward equation.

Proof. The proof follows from the same arguments for time-independent coefficients, by applying Caccioppoli's inequality to averages on local Whitney cubes $(t, 2t) \times B(x, t^{1/2})$, see Corollary 6.4. □

7.1.2 Propagators

We first recall the L^2-theory.

Proposition 7.5 (L^2-theory)**.** *Let $u_0 \in L^2$ and $F \in L^2(\mathbb{R}^{1+n}_+) \simeq T^2_0$. Then there exists a unique global weak solution u to the Cauchy problem*

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F \\ u(0) = u_0 \end{cases}$$

so that $\nabla u \in L^2(\mathbb{R}^{1+n}_+)$. Moreover, u belongs to $C([0,\infty); L^2)$ with

$$\sup_{t \geq 0} \|u(t)\|_{L^2} \leq \|u_0\|_{L^2} + (2\Lambda_0)^{-1/2} \|F\|_{L^2(\mathbb{R}^{1+n}_+)},$$

and

$$\|\nabla u\|_{L^2(\mathbb{R}^{1+n}_+)} \leq (2\Lambda_0)^{-1/2} \|u_0\|_{L^2} + \Lambda_0^{-1} \|F\|_{L^2(\mathbb{R}^{1+n}_+)}.$$

The existence is due to [Lio57], and the uniqueness in the class $\nabla u \in L^2(\mathbb{R}^{1+n}_+)$ is due to [AMP19], see [AP25, Theorem 2.2] for a survey. Moreover, [AMP19] further exploits the L^2-theory to obtain existence and uniqueness of fundamental solution operators, also called propagators. The reader can refer there for the proof of the following basic facts of propagators.

Corollary 7.6 (Propagators)**.** *There exists a family of contractions on $L^2(\mathbb{R}^n)$, $(\Gamma_A(t,s))_{0 \leq s \leq t < \infty}$, called* (forward) propagators associated to A *so that for any $h \in L^2$ and $s \geq 0$, $u(t) := \Gamma_A(t,s)h$ is the unique weak solution on $(s,\infty) \times \mathbb{R}^n$ to the Cauchy problem*

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = 0, \quad t > s \\ u(s) = h \end{cases}$$

with $\nabla u \in L^2((s,\infty) \times \mathbb{R}^n)$. Moreover, u belongs to $C_0([s,\infty); L^2)$. [1]

[1] Here, $C_0([s,\infty); E)$ is the space of continuous functions with limit 0 as $t \to \infty$ in the prescribed topology.

By reversing time, we also obtain the fundamental solution operators for the backward equation, also called backward propagators.

Corollary 7.7 (Backward propagators)**.** *There exists a family of contractions on L^2, $(\Gamma^-_{A^*}(s,t))_{0\le s\le t<\infty}$, called* backward propagators associated to A^*, *so that for any $h \in L^2$ and $t > 0$, $\widetilde{u}(s) := \Gamma^-_{A^*}(s,t)h$ is the unique weak solution on $(0,t) \times \mathbb{R}^n$ to the backward Cauchy problem*

$$\begin{cases} -\partial_s \widetilde{u} - \operatorname{div}(A(s)^* \nabla \widetilde{u}) = 0, & s < t \\ u(t) = h \end{cases}$$

so that $\nabla \widetilde{u} \in L^2((0,t) \times \mathbb{R}^n)$. Moreover, for fixed $t > 0$, it satisfies

$$\Gamma^-_{A^*}(s,t) = \Gamma_A(t,s)^* = \Gamma_{\widetilde{A}_t}(t-s,0), \quad 0 \le s \le t, \tag{7.2}$$

where

$$\widetilde{A}_t(s,x) := \begin{cases} A^*(t-s,x), & \text{if } 0 \le s \le t \\ \Lambda_0 \mathbb{I} & \text{if } s > t \end{cases}. \tag{7.3}$$

Consequently, $\widetilde{u}$ belongs to $C([0,t];L^2)$.

Remark that $\widetilde{A}_t$ has the same ellipticity as A. When $s > t$, our construction of $\widetilde{A}_t$ is different from that in [AMP19, Lemma 3.16], but the identity (7.2) still holds for $0 \le s \le t$.

The propagators provide explicit formulas for weak solutions. Define the *propagator solution map* $\mathcal{E}_A$ from L^2 to $L^\infty((0,\infty);L^2) \cap L^2_{\mathsf{loc}}((0,\infty);W^{1,2}_{\mathsf{loc}})$ by

$$\mathcal{E}_A(u_0)(t,x) := (\Gamma_A(t,0)u_0)(x), \quad t > 0, \ x \in \mathbb{R}^n. \tag{7.4}$$

Define the *Duhamel operator* $\mathcal{L}^A_1$ from $L^2(\mathbb{R}^{1+n}_+)$ to $L^\infty_{\mathsf{loc}}([0,\infty);L^2)$ by the L^2-valued Bochner integrals (verified below)

$$\mathcal{L}^A_1(f)(t) := \int_0^t \Gamma_A(t,s)f(s)ds, \quad t > 0, \tag{7.5}$$

and the *backward Duhamel operator* $(\mathcal{L}^A_1)^*$ from $L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$ to $L^\infty_{\mathsf{loc}}([0,\infty);L^2)$ by the L^2-valued Bochner integrals (also verified below)

$$(\mathcal{L}^A_1)^*(f)(s) := \int_s^\infty \Gamma_A(t,s)^* f(t)dt, \quad 0 < s < t. \tag{7.6}$$

As we shall see in Lemma 7.13, $(\mathcal{L}^A_1)^*$ is indeed the adjoint of $\mathcal{L}^A_1$ with respect to the $L^2(\mathbb{R}^{1+n}_+)$-duality.

Let us explain why the integral in (7.5) is a Bochner integral. The one in (7.6) follows similarly. We first verify the strong measurability of the function $s \mapsto \Gamma_A(t,s)f(s)$, valued in L^2. Indeed, we infer from Corollary 7.7 that for any $\phi, \psi \in L^2$, the function

$$s \mapsto \langle \Gamma_A(t,s)\phi, \psi\rangle = \langle \phi, \Gamma_A(s,t)^*\psi\rangle = \langle \phi, \Gamma^-_{A^*}(s,t)\psi\rangle \tag{7.7}$$

is continuous, hence (Borel) measurable. So for any $f \in L^2(\mathbb{R}^{1+n}_+)$, $s \mapsto \langle \Gamma_A(t,s) f(s), \psi \rangle$ is measurable. We thus get $s \mapsto \Gamma_A(t,s) f(s)$ is weakly measurable, and hence strongly measurable by Pettis' measurability theorem (see [HvNVW16, Theorem 1.1.20]), since L^2 is separable. The integrability comes from the fact that $\|\Gamma_A(t,s)\|_{\mathscr{L}(L^2)} \leq 1$.

Define the *Lions operator* $\mathcal{R}^A_{1/2}$ by the formal integral

$$\mathcal{R}^A_{1/2}(F)(t) := \int_0^t \Gamma_A(t,s) \operatorname{div} F(s) ds, \quad t > 0. \tag{7.8}$$

We do not know whether the integral in (7.8) converges in the sense of Bochner integrals on L^2, because we do not have estimates for the operator norm of $\Gamma_A(t,s) \operatorname{div}$ in $\mathscr{L}(L^2)$ for each s and t. Instead, it is only interpreted in the weak sense, *i.e.*, for any $t > 0$, $\mathcal{R}^A_{1/2}(F)(t)$ is defined as a continuous linear functional on L^2 by

$$\langle \mathcal{R}^A_{1/2}(F)(t), h \rangle_{L^2(\mathbb{R}^n)} := -\int_0^t \langle F(s), \nabla \Gamma_A(t,s)^* h \rangle_{L^2(\mathbb{R}^n)} ds,$$

for any $h \in L^2$. The integral on the right-hand side converges as

$$\begin{aligned} \int_0^t \left| \langle F(s), \nabla \Gamma_A(t,s)^* h \rangle_{L^2(\mathbb{R}^n)} \right| ds &\leq \|F\|_{L^2(\mathbb{R}^{1+n}_+)} \|\nabla \Gamma_A(t,\cdot)^* h\|_{L^2((0,t)\times\mathbb{R}^n)} \\ &\leq (2\Lambda_0)^{-1/2} \|F\|_{L^2(\mathbb{R}^{1+n}_+)} \|h\|_{L^2}. \end{aligned}$$

In the last inequality we use (7.2) and the energy inequality in Proposition 7.5 for reversed time, noting that $\widetilde{A}_t$ has the same ellipticity as A. In particular, we get $\mathcal{R}^A_{1/2}$ is bounded from $L^2(\mathbb{R}^{1+n}_+)$ to $L^\infty((0,\infty); L^2)$.

Proposition 7.8. *Let $u_0 \in L^2$, $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$, and $F \in L^2(\mathbb{R}^{1+n}_+)$.*

(i) *$v := \mathcal{L}^A_1(f)$ is a global weak solution to the inhomogeneous Cauchy problem*

$$\begin{cases} \partial_t v - \operatorname{div}(A\nabla v) = f \\ v(0) = 0 \end{cases} . \tag{IC}$$

(ii) *$w := (\mathcal{L}^A_1)^*(f)$ is a weak solution on $\mathbb{R}^{1+n}_+$ to the backward equation*

$$-\partial_s w - \operatorname{div}(A(s)^* \nabla w) = f.$$

(iii) *Let u be the unique global weak solution to the Cauchy problem*

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F \\ u(0) = u_0 \end{cases}$$

with $\nabla u \in L^2(\mathbb{R}^{1+n}_+)$. Then the following properties hold.

(1) (Duhamel's formula) $u = \mathcal{E}_A(u_0) + \mathcal{R}^A_{1/2}(F)$ *in* $\mathscr{D}'(\mathbb{R}^{1+n}_+)$.

(2) *Define* $\widetilde{F} := (A - \mathbb{I})\nabla\mathcal{R}^A_{1/2}(F) + F$ *in* $L^2(\mathbb{R}^{1+n}_+)$. *Then*

$$\mathcal{R}^A_{1/2}(F) = \mathcal{R}^{\mathbb{I}}_{1/2}(\widetilde{F}) = \operatorname{div}\mathcal{L}^{\mathbb{I}}_1(\widetilde{F}) \quad \text{in } \mathscr{D}'(\mathbb{R}^{1+n}_+).$$

(3) *It holds that*

$$\begin{aligned}\mathcal{E}_A(u_0) &= \mathcal{E}_{\mathbb{I}}(u_0) + \mathcal{R}^A_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_{\mathbb{I}}(u_0)) \\ &= \mathcal{E}_{\mathbb{I}}(u_0) + \mathcal{R}^{\mathbb{I}}_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_A(u_0)).\end{aligned}$$

Proof. The statement (i) directly follows from [AE23b, Theorem 2.54]. Applying the same argument to the backward equation gives (ii), thanks to (7.2). To prove (iii) (1), we infer from [AE23b, Theorem 2.54] that Duhamel's formula holds in $L^\infty((0,T);L^2)$ for any $T>0$, hence in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$. The statements (2) and (3) follow from the same arguments as in Corollary 6.7. Details are left to the reader. □

To end this section, we exhibit the quantitative L^p-theory of the propagators by employing the notion of off-diagonal decay, see Definition 4.1. Denote by Δ the set $\{(t,s) \in (0,\infty)\times(0,\infty) : t = s\}$ and write $\Delta^c := ((0,\infty)\times(0,\infty))\setminus\Delta$. We follow the notation $[p,q] := \frac{1}{p} - \frac{1}{q}$ for any $p,q \in (0,\infty]$, if there is no confusion with closed intervals.

Definition 7.9 (Off-diagonal decay). Let $1 \le p \le q \le \infty$. Let $\{K(t,s)\}_{(t,s)\in\Delta^c}$ be a family of bounded operators on $L^2(\mathbb{R}^n)$. We say $\{K(t,s)\}$ satisfies the $L^p - L^q$ *off-diagonal decay* if there are constants $c, C > 0$ so that for any $t > s$, $E, F \subset \mathbb{R}^n$ as Borel sets, and $f \in L^2 \cap L^p$,

$$\|\mathbb{1}_E K(t,s)\mathbb{1}_F f\|_{L^q} \le C(t-s)^{-\frac{n}{2}[p,q]} \exp\left(-c\frac{\operatorname{dist}(E,F)^2}{t-s}\right)\|\mathbb{1}_F f\|_{L^p}.$$

Proposition 7.10 (L^p-theory of propagators). *Let $A \in L^\infty(\mathbb{R}^{1+n}_+;\mathrm{Mat}_n(\mathbb{C}))$ be uniformly elliptic. Let $(\Gamma_A(t,s))$ be the propagators associated to A. Then the following properties hold.*

(i) *There exist $p_-(A) \in [1,2)$ and $p_+(A) \in (2,\infty]$ so that $(p_-(A), p_+(A))$ is the maximal open set of exponents $p \in [1,\infty]$ for which $(\Gamma_A(t,s))$ is uniformly bounded on L^p.*

(ii) *For $p_-(A) < p \le 2$, $(\Gamma_A(t,s))$ has $L^p - L^2$ off-diagonal decay. For $2 \le q < p_+(A)$, $(\Gamma_A(t,s))$ has $L^2 - L^q$ off-diagonal decay.*

Proof. First consider (i). Since $(\Gamma_A(t,s))$ is a family of contractions on L^2, particularly, it is uniformly bounded on L^2. If $(\Gamma_A(t,s))$ is uniformly bounded on L^q for some $q \ne 2$, then by interpolation, one gets $(\Gamma_A(t,s))$ is uniformly bounded on L^p for all p between q and 2. So all the $p \in [1,\infty]$ for which $(\Gamma_A(t,s))$ is uniformly bounded on L^p form an interval containing 2. Let $p_-(A)$ and $p_+(A)$ be the left and right extremes of this interval. It has been shown in [Zat20, Theorem 1.6] that there exists $\epsilon > 0$ only depending on n and the ellipticity of A so that $(\Gamma_A(t,s))$ is uniformly bounded on L^p for $2-\epsilon < p < 2+\epsilon$. We thus infer that $p_-(A) < 2$ and $p_+(A) > 2$. This proves (i).

The second statement (ii) combines [AMP19, Lemmas 4.9 and 4.11]. This completes the proof. □

For time-independent coefficients, the critical numbers $p_\pm(A)$ coincide with $p_-^\flat(L)$ and $p_+(L)$, respectively, where $L := -\operatorname{div}(A(x)\nabla)$, see Section 6.1.3. We also know the inequalities $p_-^\flat(L) < \frac{2n}{n+2}$ and $p_+(L) > \frac{2n}{n-2}$ are best possible. However, for time-dependent coefficients, we do not know whether the bounds $p_-(A) < 2$ and $p_+(A) > 2$ are sharp.

7.2 Inhomogeneous Cauchy problem

This section is concerned with the existence of weak solutions to the inhomogeneous Cauchy problem

$$\begin{cases} \partial_t v - \operatorname{div}(A\nabla v) = f \\ v(0) = 0 \end{cases} . \tag{IC}$$

The weak solutions are constructed by the extension of the Duhamel operator $\mathcal{L}_1^A$ defined in (7.5). We collect the properties of the extension in the following theorem as a general result. Part of the proof is deferred to the end of next section. Let us introduce the number

$$p_A(\beta) := \frac{np_-(A)}{n+(2\beta+1)p_-(A)}.$$

For time-independent coefficients, it agrees with $p_L^\flat(\beta)$ defined in (6.11).

Theorem 7.11 (Extension of $\mathcal{L}_1^A$)**.** *Let $\beta > -1/2$ and $p_A(\beta) < p \le \infty$. Then $\mathcal{L}_1^A$ extends to a bounded operator from T^p_β to $T^p_{\beta+1}$, also denoted by $\mathcal{L}_1^A$. Moreover, the following properties hold for any $f \in T^p_\beta$ and $v := \mathcal{L}_1^A(f)$.*

(a) (Regularity) *v lies in $T^p_{\beta+1}$ and ∇v lies in $T^p_{\beta+1/2}$ with*

$$\|v\|_{T^p_{\beta+1}} \lesssim \|f\|_{T^p_\beta}, \quad \|\nabla v\|_{T^p_{\beta+1/2}} \lesssim \|f\|_{T^p_\beta}.$$

(b) *v is a global weak solution to $\partial_t v - \operatorname{div}(A\nabla v) = f$.*

(c) (Explicit formula) *It holds that*

$$v = \mathcal{L}_1^{\mathbb{I}}(f) + \mathcal{R}_{1/2}^{\mathbb{I}}((A - \mathbb{I})\nabla v). \tag{7.9}$$

(d) (Continuity and trace) $v \in C([0,\infty); \mathscr{S}')$ *with* $v(0) = 0$*. As* $t \to 0$*, the convergence also occurs in*

$$\begin{cases} L^p & \text{if } p_A(\beta) < p \leq 2 \\ E^q_\delta & \text{if } 2 < p \leq \infty \end{cases}, \tag{7.10}$$

where $\delta > 0$ *and* $q \in [p, \infty]$ *are arbitrary parameters.*

Consequently, v *is a global weak solution to* (IC) *with source term* f*.*

Remark 7.12. For time-independent coefficients, the range of β and p in Theorem 7.11 coincides with that in Theorem 6.11. There, we also establish the maximal regularity estimates, *i.e.*, both $\partial_t u$ and $\operatorname{div}(A\nabla u)$ lie in T^p_β with $\|\partial_t u\|_{T^p_\beta} + \|\operatorname{div}(A\nabla u)\|_{T^p_\beta} \lesssim \|f\|_{T^p_\beta}$. For time-dependent coefficients, it is not clear, since we do not have appropriate estimates for the operator $h \mapsto \operatorname{div}(A\nabla\Gamma(t,s)h)$.

In this section, we prove (a) and (b). The proof of (c) and (d) is postponed to Section 7.3.4. We employ the machinery of singular integral operators on tent spaces developed in Chapter 4. The following lemma verifies that both $\mathcal{L}_1^A$ and the backward Duhamel operator $(\mathcal{L}_1^A)^*$ defined in (7.6) are involved in this framework.

Lemma 7.13. *The operator* $\mathcal{L}_1^A$ *(resp.* $(\mathcal{L}_1^A)^*$*) belongs to* $\mathrm{SIO}^{1+}_{2,q,\infty}$ *(resp.* $\mathrm{SIO}^{1+}_{2,q',\infty}$*) for* $p_-(A) < q < p_+(A)$*.*

Proof. For $\mathcal{L}_1^A$, it suffices to prove the kernel $K(t,s) := \mathbb{1}_{\{t>s\}}(t,s)\Gamma_A(t,s)$ belongs to $\mathrm{SK}^1_{2,q,\infty}$, see Definitions 4.2 and 4.9.

First, we show K is strongly measurable, *i.e.*, for any $\phi \in L^2$, the function $(t,s) \mapsto K(t,s)\phi$ is strongly measurable, valued in L^2. Let $\psi \in L^2$. We have shown in (7.7) that $s \mapsto \langle \Gamma_A(t,s)\phi, \psi\rangle$ is continuous, and Corollary 7.6 implies $t \mapsto \Gamma_A(t,s)\phi$ is continuous valued in L^2, so $t \mapsto \langle \Gamma_A(t,s)\phi, \psi\rangle$ is continuous. Thus, $(t,s) \mapsto \langle \Gamma_A(t,s)\phi, \psi\rangle$ is separately continuous on $\{t > s\}$. This implies $(t,s) \mapsto \Gamma_A(t,s)\phi = K(t,s)\phi$ is weakly measurable, and the strong measurability thus follows by Pettis' measurability theorem and the fact that L^2 is separable.

Next, since $(\Gamma_A(t,s))$ is a family of contractions on L^2, we have that $\|K(t,s)\|_{\mathscr{L}(L^2)} \leq 1$.

Finally, the off-diagonal decay of K comes from Proposition 7.10 (ii). This proves the kernel K belongs to $\mathrm{SK}^1_{2,q,\infty}$, and concludes for $\mathcal{L}_1^A$.

For $(\mathcal{L}_1^A)^*$, it follows by duality, see Proposition 4.11. □

Let us prove Theorem 7.11 (a) and (b).

Proof of Theorem 7.11 (a) and (b). First consider (a). The bounded extension of $\mathcal{L}_1^A$ is a direct consequence of Proposition 4.12 and Corollary 4.14, where the conditions are verified in Lemma 7.13. To prove the gradient estimates, we first pick $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$. Proposition 7.8 (i) says $v = \mathcal{L}_1^A(f)$ is a global weak solution to $\partial_t v - \operatorname{div}(A\nabla v) = f$. So Corollary 7.4 yields

$$\|\nabla v\|_{T^p_{\beta+1/2}} \lesssim \|v\|_{T^p_{\beta+1}} + \|f\|_{T^p_\beta} \lesssim \|f\|_{T^p_\beta}.$$

Then a density argument extends the above estimates to all $f \in T^p_\beta$ (or weak*-density if $p = \infty$). This proves (a).

For (b), we know from (a) that $v := \mathcal{L}_1^A(f)$ lies in $L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$, since all the tent spaces embed into $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$. Moreover, for $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$, v satisfies $\partial_t v - \operatorname{div}(A\nabla v) = f$ in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$. Using the boundedness of $\mathcal{L}_1^A$ and $\nabla \mathcal{L}_1^A$, one can extend this identity (valued in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$) to all $f \in T^p_\beta$ by density, or weak*-density if $p = \infty$. □

Corollary 7.14. *Let $\beta > -1/2$ and $1 \leq p' < \max\{p_A(\beta), 1\}'$. Then $(\mathcal{L}_1^A)^*$ extends to a bounded operator from $T^{p'}_{-\beta-1}$ to $T^{p'}_{-\beta}$, also denoted by $(\mathcal{L}_1^A)^*$. It additionally satisfies*

$$\|\nabla(\mathcal{L}_1^A)^*(f)\|_{T^{p'}_{-\beta-1/2}} \lesssim \|f\|_{T^{p'}_{-\beta-1}}.$$

Proof. The statements follow from adapting the arguments in the proof of Theorem 7.11 (a) to backward singular integral operators (see Proposition 4.13 and Corollary 4.15) and backward equations. We only need to note that for $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$, Proposition 7.8 (ii) says $w := (\mathcal{L}_1^A)^*(f)$ is a weak solution to the backward equation $-\partial_s w - \operatorname{div}(A(s)^*\nabla w) = f$. Detailed verification is left to the reader. □

7.3 Lions' equation

In this section, we prove existence of weak solutions to the *Lions equation*

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F \\ u(0) = 0 \end{cases}, \tag{L}$$

by extension of the Lions operator $\mathcal{R}^A_{1/2}$ defined in (7.8). The main theorem of this section summarizes the properties of the extension.

To precise our results, we introduce some exponents. Let $\zeta < -1/2$ be a fixed reference number. Define the exponents $p^\pm_\zeta(\beta)$ by

$$p^-_\zeta(\beta) := \begin{cases} \frac{2(2\zeta+1)p_-(A)}{4(\zeta-\beta)+(2\beta+1)p_-(A)} & \text{if } \zeta \leq \beta < -1/2 \\ p_A(\beta) & \text{if } \beta \geq -1/2 \end{cases}, \tag{7.11}$$

and

$$p_\zeta^+(\beta) := \begin{cases} \frac{2(2\zeta+1)}{2\beta+1} & \text{if } \zeta \le \beta < -1/2 \\ +\infty & \text{if } \beta \ge -1/2 \end{cases}. \tag{7.12}$$

Note that $p_\zeta^-(\zeta) = p_\zeta^+(\zeta) = 2$. To illustrate these exponents, we give graphic representations in Figure 7.1. In this figure, we also write p for $1/p$ to ease the presentation. We use red dashed line for the graph of $p_\zeta^-(\beta)$ for $\beta < -1/2$, red normal line for that of $p_\zeta^+(\beta)$, and blue dashed line for that of $p_A(\beta)$. Parallel lines to the blue dashed line are lines of embedding for Hardy–Sobolev spaces and weighted tent spaces going downward.

We shall introduce a new parameter $\beta_A \in [-1, -1/2)$, which only depends on the ellipticity of A and the dimension n, as the lower bound of β. Taking $\zeta = \beta_A$, the orange shaded trapezoid becomes the region of well-posedness for $\dot{H}^{2\beta+1,p}$-initial data, while the blue shaded is for constant initial data.

In particular, consider the orange triangle below the line $\beta = -1/2$. We have stated things in a way that it is above two segments. Actually, this is an artifact of our statements made to simplify the exposition. One could state things in a way that this orange triangle is above some convex curve that passes through $(0, -1/2)$, $(1/2, \beta_A)$, and $(1/p_-(A), -1/2)$. We do not know what this precise curve should look like. In fact, for $\beta < -1/2$, the red normal line for $p_{\beta_A}^+(\beta)$ is only due to our choice of β_A. If instead we take the lower bound and if it is not attained, then this line should be dashed, but the point $(0, -1/2)$ is always included.

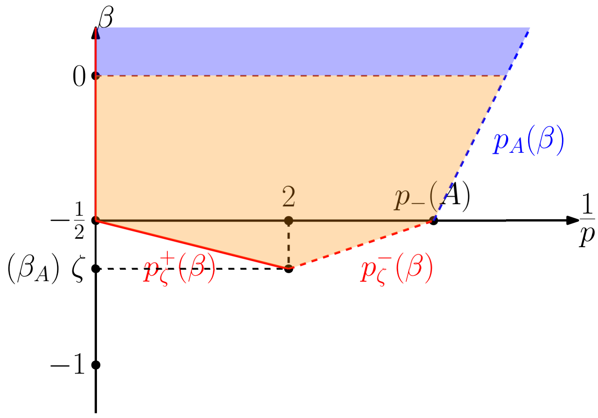


Figure 7.1: Region of well-posedness

Theorem 7.15 (Extension of $\mathcal{R}_{1/2}^A$)**.** *There exists $\beta_A \in [-1, -1/2)$ only depending on the ellipticity of A and the dimension n so that for $\beta > \beta_A$ and $p_{\beta_A}^-(\beta) < p \le p_{\beta_A}^+(\beta)$, $\mathcal{R}_{1/2}^A$ extends to a bounded operator from $T_{\beta+1/2}^p$ to $T_{\beta+1}^p$, also denoted by $\mathcal{R}_{1/2}^A$. Moreover, the following properties hold for any $F \in T_{\beta+1/2}^p$ and $u := \mathcal{R}_{1/2}^A(F)$.*

(a) (Regularity) *u lies in* $T^p_{\beta+1}$ *and* ∇u *lies in* $T^p_{\beta+1/2}$ *with*

$$\|u\|_{T^p_{\beta+1}} \lesssim \|F\|_{T^p_{\beta+1/2}}, \quad \|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|F\|_{T^p_{\beta+1/2}}.$$

(b) *u is a global weak solution to* $\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F$.

(c) (Explicit formula) *Define* $\widetilde{F} := (A - \mathbb{I})\nabla u + F$. *Then*

$$u = \mathcal{R}^{\mathbb{I}}_{1/2}(\widetilde{F}) = \operatorname{div} \mathcal{L}^{\mathbb{I}}_1(\widetilde{F}) \quad \text{in } \mathscr{D}'(\mathbb{R}^{1+n}_+). \tag{7.13}$$

(d) (Continuity and traces) $u \in C([0,\infty); \mathscr{S}')$ *with* $u(0) = 0$. *As* $t \to 0$, *the convergence also occurs in the following spaces shown in Table 7.1, with arbitrary parameters* $\delta > 0$, $q \in [p,\infty]$, *and* $s \in [-1, 2\beta+1]$.

Table 7.1: Trace spaces for $\mathcal{R}^A_{1/2}(F)$.

Conditions	$p_A(\beta) < p \le 2$	$2 < p < \infty$	$p = \infty$
$\beta > -1/2$	L^p	$E^q_\delta \cap E^{-1,q}_\delta$	$E^\infty_\delta \cap E^{-1,\infty}_\delta$
$\beta = -1/2$	L^p	$E^q_\delta \cap E^{-1,q}_\delta$	$L^2_{\mathsf{loc}} \cap E^{-1,\infty}_\delta$
$\beta_A < \beta < -1/2$	$\dot{H}^{s,p}$	$E^{-1,q}_\delta$	not applicable

Consequently, u is a global weak solution to the Lions equation (L).

Remark 7.16. For time-independent coefficient matrices, it has been shown in Theorem 6.22 that $\beta_A = -1$, and the range of p for which these properties are valid, strictly contains $p^-_{-1}(\beta) < p \le \infty$. But here, we present new trace spaces E^q_δ for $\beta \ge -1/2$ and $2 < p \le \infty$, which will be useful for proving uniqueness.

Remark 7.17. As we shall see in the proof, for $\beta = -1/2$ and $p = \infty$, we still obtain that for any $\delta > 0$, $\sup_{0<t<\delta} \|u(t)\|_{E^\infty_\delta}$ is uniformly bounded, although we do not know how to prove the trace in E^∞_δ.

Remark 7.18. In the range of $\beta > -1$ and $\frac{n}{n+2\beta+2} < p \le \infty$, if one can show that $\mathcal{R}^A_{1/2}$ extends to a bounded operator from $T^p_{\beta+1/2}$ to $T^p_{\beta+1}$, then all subsequent properties can be proved by the same arguments. Moreover, for $p = \infty$, if $\mathcal{R}^A_{1/2}$ extends to a bounded operator from $T^\infty_{\beta+1/2}$ to $T^\infty_{\beta+1}$ for some $\beta \in (-1,-1/2)$, then for any $F \in T^\infty_{\beta+1/2}$, $\mathcal{R}^A_{1/2}(F)(t)$ tends to 0 in $E^{-1,\infty}_\delta$ as $t \to 0$.

Remark 7.19. Let us compare with the recent work [AP25]. Their work treats the case $\beta = -1/2$ and $2 \le p \le \infty$ (and an extra case $1 - \epsilon < p < 1$ for real coefficients). Our results furnish the cases $p_-(A) < p < 2$ and treat new cases $\beta \ne -1/2$. We also precise the continuity, particularly the trace spaces.

The proof of this theorem is presented in Section 7.3.3. We first prove three lemmas on the bounded extension of $\mathcal{R}^A_{1/2}$ in different ranges.

Lemma 7.20. *Let $\beta > -1/2$ and $p_A(\beta) < p \le \infty$. Then $\mathcal{R}^A_{1/2}$ extends to a bounded operator from $T^p_{\beta+1/2}$ to $T^p_{\beta+1}$.*

Lemma 7.21. *There exists $\epsilon > 0$ only depending on the ellipticity of A and the dimension n so that the properties of $\mathcal{R}^A_{1/2}$ in Lemma 7.20 are also valid for $-1/2 - \epsilon < \beta < -1/2 + \epsilon$ and $p = 2$.*

Lemma 7.22. *The properties of $\mathcal{R}^A_{1/2}$ in Lemma 7.20 are also valid for $\beta = -1/2$ and $p = \infty$.*

The proofs are provided right below in Sections 7.3.1, 7.3.2, and 7.3.3.

7.3.1 Proof of Lemma 7.20

Let $\beta > -1/2$ and $p_A(\beta) < p \le \infty$. We split the proof into two cases.

Case 1: $\max\{p_A(\beta), 1\} < p \le \infty$ We argue by duality. Let $F \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$ and $\psi \in C^\infty_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$. Fubini's theorem yields

$$\begin{aligned}\langle \mathcal{R}^A_{1/2}(F), \psi\rangle_{L^2(\mathbb{R}^{1+n}_+)} &= -\int_0^\infty \int_{\mathbb{R}^n} F(s,y) \left(\int_s^\infty \overline{\nabla \Gamma_A(t,s)^* \psi(t)}(y) dt \right) dsdy \\ &= -\langle F, \nabla (\mathcal{L}^A_1)^*(\psi)\rangle_{L^2(\mathbb{R}^{1+n}_+)}.\end{aligned}$$

We further apply duality of tent spaces and Corollary 7.14 to get

$$|\langle \mathcal{R}^A_{1/2}(F), \psi\rangle_{L^2(\mathbb{R}^{1+n}_+)}| \lesssim \|F\|_{T^p_{\beta+1/2}} \|\nabla (\mathcal{L}^A_1)^* \psi\|_{T^{p'}_{-\beta-1/2}} \lesssim \|F\|_{T^p_{\beta+1/2}} \|\psi\|_{T^{p'}_{-\beta-1}}.$$

The bounded extension of $\mathcal{R}^A_{1/2}$ hence follows by density, or weak*-density if $p = \infty$.

Case 2: $p_A(\beta) < p \le 1$ We use atomic decomposition of tent spaces. Recall that for $\beta \in \mathbb{R}$ and $0 < p \le 1$, a function $a \in L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$ is called a $T^p_{\beta+1/2}$*-atom*, if there exists a ball $B \subset \mathbb{R}^n$ so that $\operatorname{supp}(a) \subset [0, r(B)^2] \times B$, and

$$\|a\|_{L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)} \le |B|^{-[p,2]}.$$

The ball B is called *associated* to a. Write $r := r(B)$, $C_0 := 2B$, and $C_j := 2^{j+1}B \setminus 2^j B$ for $j \ge 1$. For $j \ge 4$, define

$$\begin{aligned} M^{(1)}_j &:= \left(0, (2^3 r)^2\right] \times C_j, \\ M^{(2)}_j &:= \left((2^3 r)^2, (2^j r)^2\right) \times C_j, \\ M^{(3)}_j &:= \left[(2^j r)^2, (2^{j+1} r)^2\right) \times 2^{j+1} B. \end{aligned}$$

The following lemma describes the molecular decay of $\mathcal{R}^A_{1/2}$ acting on atoms.

Lemma 7.23 (Molecular decay)**.** *Let $\beta > -1/2$, $0 < p \leq 1$, and $p_-(A) < q < 2$. There exists a constant $c > 0$ depending on A and q so that for any $T^p_{\beta+1/2}$-atom a with an associated ball $B \subset \mathbb{R}^n$, the following estimates hold for $u := \mathcal{R}^A_{1/2}(a)$ and $j \geq 4$,*

$$\|\mathbb{1}_{M_j^{(1)}} u\|_{L^2_{\beta+1}(\mathbb{R}^{1+n}_+)} \lesssim 2^{jn[p,2]} e^{-c2^{2j}} |2^{j+1}B|^{-[p,2]}, \tag{7.14}$$

$$\|\mathbb{1}_{M_j^{(2)}} u\|_{L^2_{\beta+1}(\mathbb{R}^{1+n}_+)} \lesssim 2^{-j(2\beta+1+n[q,p])} |2^{j+1}B|^{-[p,2]}, \tag{7.15}$$

$$\|\mathbb{1}_{M_j^{(3)}} u\|_{L^2_{\beta+1}(\mathbb{R}^{1+n}_+)} \lesssim 2^{-j(2\beta+1+n[q,p])} |2^{j+1}B|^{-[p,2]}. \tag{7.16}$$

The proof is provided right below. Admitting this lemma, let us prove the bounded extension of $\mathcal{R}^A_{1/2}$. Thanks to Corollary 3.6, it suffices to verify $\mathcal{R}^A_{1/2}$ is uniformly bounded on $T^p_{\beta+1/2}$-atoms. Let a be a $T^p_{\beta+1/2}$-atom, $B \subset \mathbb{R}^n$ be a ball associated to a, and $u := \mathcal{R}^A_{1/2}(a)$. Write $Q_0 := \big(0, (2^3 r(B))^2\big) \times 2^3 B$. We get from Case 1 that

$$\|\mathbb{1}_{Q_0} u\|_{L^2_{\beta+1}(\mathbb{R}^{1+n}_+)} \lesssim \|a\|_{L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)} \lesssim |2^3 B|^{-[p,2]}. \tag{7.17}$$

Meanwhile, pick $q < 2$ sufficiently close to $p_-(A)$ so that

$$p_A(\beta) = \frac{np_-(A)}{n + (2\beta+1)p_-(A)} < \frac{nq}{n+(2\beta+1)q} < p \leq 1.$$

In particular, we get $2\beta+1+n[q,p] > 0$. Applying Lemma 7.23 to such q gives estimates of u on molecules $M_j^{(i)}$ for $1 \leq i \leq 3$ and $j \geq 4$. These estimates, together with (7.17), yield

$$\|u\|^p_{T^p_{\beta+1}} \lesssim 1 + \sum_{j\geq 4} 2^{jn[p,2]p} e^{-c2^{2j}p} + \sum_{j \geq 4} 2^{-j(2\beta+1+n[q,p])p} \leq C < \infty.$$

Here, we use the atomic decomposition of tent spaces, see Proposition 3.5 or [CMS85, Proposition 5]. This completes the proof of Lemma 7.20.

Proof of Lemma 7.23. The proof of (7.15) and (7.16) is a verbatim adaptation of Lemma 6.30, (6.41) and (6.42), using the $L^2 - L^{q'}$ off-diagonal decay of $(\Gamma_A(t,s)^*)$. Let us focus on (7.14). As $a \in L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)$, we show in Case 1 that $u := \mathcal{R}^A_{1/2}(a)$ lies in $L^2_{\beta+1}(\mathbb{R}^{1+n}_+)$. Let $\phi \in C^\infty_c(\mathbb{R}^n)$ with $\operatorname{supp}(\phi) \subset C_j$. We have

$$\langle u(t), \phi \rangle_{L^2(\mathbb{R}^n)} = -\int_0^t \int_{\mathbb{R}^n} a(s,y) \overline{\nabla \Gamma_A(t,s)^* \phi}(y) ds dy.$$

Write $v_t(s,y) := \Gamma_A(t,s)^*\phi(y)$. As $2\beta+1>0$, applying the properties of a and Cauchy–Schwarz inequality gives

$$\begin{aligned}\left|\langle u(t),\phi\rangle_{L^2(\mathbb{R}^n)}\right|^2 &\le \|a\|^2_{L^2_{\beta+1/2}(\mathbb{R}^{1+n}_+)}\int_0^t\int_B s^{2\beta+1}|\nabla v_t(s,y)|^2 dsdy\\ &\le |B|^{-2[p,2]}t^{2\beta+1}\int_0^t\int_B|\nabla v_t(s,y)|^2dsdy.\end{aligned}$$

Then we take a covering of $(0,t)\times B$ by reverse Whitney cubes $((1-2^{-k-1})t,(1-2^{-k})t)\times B(x_0,(1-2^{-k-1})^{1/2}t^{1/2})$ for $k\ge 0$. Corollary 7.7 implies v_t is a weak solution to the backward equation $-\partial_s v_t - \operatorname{div}(A(s)^*\nabla v_t)=0$. By applying the Caccioppoli's inequality to each Whitney cube and summing the results, we obtain

$$\int_0^t\int_B|\nabla v_t(s,y)|^2dsdy \lesssim \int_0^t\int_{15B}\frac{|v_t(s,y)|^2}{t-s}dsdy.$$

The L^2-L^2 off-diagonal estimates of $(\Gamma_A(t,s)^*)$ provide us with a constant $c>0$ so that

$$\int_0^t\int_{15B}\frac{|v_t(s,y)|^2}{t-s}dsdy \lesssim \int_0^t \frac{1}{t-s}e^{-\frac{2c(2^jr)^2}{t-s}}\|\phi\|^2_{L^2}ds \lesssim e^{-\frac{c(2^jr)^2}{t}}\|\phi\|^2_{L^2}.$$

Gathering these estimates, by duality, we obtain

$$\|u(t)\|^2_{L^2(C_j)} \lesssim |B|^{-2[p,2]}t^{2\beta+1}e^{-\frac{c(2^jr)^2}{t}}, \quad 0<t\le(2^3r)^2.$$

Integrating both sides over $0<t\le(2^3r)^2$ gives (7.14) as desired. $\square$

7.3.2 Proof of Lemma 7.21

We use an abstract argument relying on Sneiberg's lemma. Let $\beta\in\mathbb{R}$ and $1<p<\infty$. Define

$$S^p_{\beta+1/2} := \left\{u\in L^2_{\mathsf{loc}}((0,\infty);W^{1,2}_{\mathsf{loc}}) : \nabla u\in T^p_{\beta+1/2},\ \partial_t u\in \operatorname{div}T^p_{\beta+1/2}\right\},$$

endowed with the semi-norm

$$\|u\|_{S^p_{\beta+1/2}} := \|\nabla u\|_{T^p_{\beta+1/2}} + \|\partial_t u\|_{\operatorname{div}T^p_{\beta+1/2}}.$$

Lemma 7.24. *Let $-1<\beta<0$ and $1<p<\infty$. Let $u\in S^p_{\beta+1/2}$. Then u belongs to $C([0,\infty);\mathscr{S}')$. Moreover, $u(0)$ belongs to $\dot{H}^{2\beta+1,p}+\mathbb{C}$ with*

$$\|u(0)\|_{\dot{H}^{2\beta+1,p}/\mathbb{C}} \lesssim \|u\|_{S^p_{\beta+1/2}}.$$

We call $u(0)$ the trace *of u, denoted by* $\operatorname{Tr}(u)$.

Proof. Let $u \in S^p_{\beta+1/2}$. As $\partial_t u - \Delta u \in \operatorname{div} T^p_{\beta+1/2}$, Theorem 6.22 says there is a unique weak solution $v \in S^p_{\beta+1/2}$ to the Cauchy problem

$$\begin{cases} \partial_t v - \Delta v = \partial_t u - \Delta u \\ v(0) = 0 \end{cases}.$$

It additionally belongs to $C([0,\infty);\mathscr{S}')$ and satisfies

$$\|v\|_{S^p_{\beta+1/2}} \lesssim \|\partial_t u - \Delta u\|_{\operatorname{div} T^p_{\beta+1/2}} \lesssim \|u\|_{S^p_{\beta+1/2}}.$$

Observe that $w := u - v$ lies in $S^p_{\beta+1/2}$ and w is a weak solution to the heat equation $\partial_t w - \Delta w = 0$. Then we invoke Theorem 5.17 to get that there is a unique $u_0 \in \mathscr{S}'$ so that $w = \mathcal{E}_{\mathbb{I}}(u_0)$, and

$$\begin{aligned} \|u_0\|_{\dot{H}^{2\beta+1,p}/\mathbb{C}} \eqsim \|\nabla w\|_{T^p_{\beta+1/2}} &\le \|\nabla u\|_{T^p_{\beta+1/2}} + \|\nabla v\|_{T^p_{\beta+1/2}} \\ &\le \|\nabla u\|_{T^p_{\beta+1/2}} + \|v\|_{S^p_{\beta+1/2}} \lesssim \|u\|_{S^p_{\beta+1/2}}. \end{aligned}$$

Note that for any $t > 0$, $w(t) = \mathcal{E}_{\mathbb{I}}(u_0)(t) = e^{t\Delta}u_0$, so w also belongs to $C([0,\infty);\mathscr{S}')$ with $w(0) = u_0$. Therefore, we get $u \in C([0,\infty);\mathscr{S}')$ with $u(0) = u_0$. This completes the proof. $\square$

Proposition 7.25. *Let $-1 < \beta < 0$ and $1 < p < \infty$. The map*

$$\begin{aligned} \Phi : S^p_{\beta+1/2} &\to \operatorname{div} T^p_{\beta+1/2} \times (\dot{H}^{2\beta+1,p} + \mathbb{C}) \\ \Phi(u) &:= (\partial_t u - \Delta u, \qquad \operatorname{Tr}(u)). \end{aligned}$$

is an isomorphism of semi-normed spaces. Hence, the quotient map of Φ (induced by the canonical projection modulo constants) is an isomorphism of Banach spaces from $S^p_{\beta+1/2}/\mathbb{C}$ to $\operatorname{div} T^p_{\beta+1/2} \times \dot{H}^{2\beta+1,p}/\mathbb{C}$.

Proof. Lemma 7.24 shows Φ is bounded. Let us construct the inverse Ψ. For any $g \in \operatorname{div} T^p_{\beta+1/2}$ and $v_0 \in \dot{H}^{2\beta+1,p} + \mathbb{C}$, let $v =: \Psi(g, v_0)$ be the unique weak solution in $S^p_{\beta+1/2}$ to the Cauchy problem

$$\begin{cases} \partial_t v - \Delta v = g \\ v(0) = v_0 \end{cases}. \tag{7.18}$$

The existence and uniqueness are ensured by Theorems 6.22 and 6.33. We also infer that

$$\|\Psi(g,v_0)\|_{S^p_{\beta+1/2}} \lesssim \|g\|_{\operatorname{div} T^p_{\beta+1/2}} + \|v_0\|_{\dot{H}^{2\beta+1,p}/\mathbb{C}},$$

so Ψ is bounded. Let us verify that Ψ is the inverse of Φ. First, for any $g \in \operatorname{div} T^p_{\beta+1/2}$ and $v_0 \in \dot{H}^{2\beta+1,p} + \mathbb{C}$, the equation (7.18) shows $\partial_t \Psi(g, v_0) - \Delta\Psi(g, v_0) = g$, and $\Psi(g, v_0)(t)$ converges to v_0 as $t \to 0$ in $\mathscr{D}'$, so v_0 coincides with $\mathrm{Tr}(\Psi(g, v_0))$ constructed in Lemma 7.24. We hence get $\Phi \circ \Psi(g, v_0) = (g, v_0)$.

On the other hand, for any $u \in S^p_{\beta+1/2}$, note that both u and $w := \Psi \circ \Phi(u)$ are weak solutions in $S^p_{\beta+1/2}$ to the Cauchy problem

$$\begin{cases} \partial_t w - \Delta w = \partial_t u - \Delta u \\ w(0) = \mathrm{Tr}(u) \end{cases} . \tag{7.19}$$

Thanks to the uniqueness of weak solutions to (7.19) in $S^p_{\beta+1/2}$, we get $u = w = \Psi \circ \Phi(u)$. This proves Ψ is the inverse of Φ.

The last point follows from the fact that $\Phi(c) = (0, c)$ for any constant function c. This completes the proof. □

Let us finish the proof of Lemma 7.21.

Proof of Lemma 7.21. Let $-1 < \beta < 0$ and $p = 2$. Define the map

$$\begin{aligned} \Phi_A : S^2_{\beta+1/2} &\to \operatorname{div} T^2_{\beta+1/2} \times (\dot{H}^{2\beta+1,2} + \mathbb{C}) \\ \Phi_A(u) &:= (\partial_t u - \operatorname{div}(A\nabla u), \quad \mathrm{Tr}(u)). \end{aligned} \tag{7.20}$$

We first show Φ_A is an isomorphism of semi-normed spaces for $\beta = -1/2$. Indeed, let $g \in \operatorname{div} T^2_0$ and $v_0 \in \dot{H}^{0,2} + \mathbb{C} \simeq L^2 + \mathbb{C}$. Let $v =: \Psi_A(g, v_0)$ be the unique weak solution in S^2_0 to the Cauchy problem

$$\begin{cases} \partial_t v - \operatorname{div}(A\nabla v) = g \\ v(0) = v_0 \end{cases} \tag{7.21}$$

given by Proposition 7.5. It also satisfies $\|v\|_{S^2_0} \lesssim \|g\|_{\operatorname{div} T^2_0} + \|v_0\|_{\dot{H}^{0,2}/\mathbb{C}}$, so Ψ_A is bounded. Ensured by the uniqueness of weak solutions to (7.21) in S^2_0, one can get Ψ_A is the inverse of Φ_A by adapting the proof of Proposition 7.25 *mutatis mutandis*.

Next, observe that for any $F \in T^2_0$,

$$\Phi_A^{-1}(\operatorname{div} F, 0) = \mathcal{R}^A_{1/2}(F). \tag{7.22}$$

Indeed, both $u := \Phi_A^{-1}(\operatorname{div} F, 0)$ and $v := \mathcal{R}^A_{1/2}(F)$ are weak solutions in S^2_0 to the Cauchy problem

$$\begin{cases} \partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F \\ u(0) = 0 \end{cases} . \tag{7.23}$$

Then (7.22) follows from uniqueness of weak solutions to (7.23) in S^2_0.

We claim that there is a constant $\epsilon > 0$ only depending on the ellipticity of A and the dimension n so that Φ_A is an isomorphism of semi-normed spaces for

$-1/2-\epsilon < \beta < -1/2+\epsilon$. Suppose it holds. Then the map $F \mapsto \nabla\Phi_A^{-1}(\operatorname{div} F, 0)$ extends to $T^2_{\beta+1/2}$ with

$$\begin{aligned}\|\nabla\Phi_A^{-1}(\operatorname{div} F,0)\|_{T^2_{\beta+1/2}} &\le \|\Phi_A^{-1}(\operatorname{div} F,0)\|_{S^2_{\beta+1/2}} \\ &\lesssim \|\operatorname{div} F\|_{\operatorname{div} T^2_{\beta+1/2}} \le \|F\|_{T^2_{\beta+1/2}}.\end{aligned} \tag{7.24}$$

Moreover, for any $F \in T^2_{\beta+1/2} \cap T^2_0$, Proposition 7.8 (2) says

$$u = \mathcal{R}^A_{1/2}(F) = \mathcal{R}^{\mathbb{I}}_{1/2}((A-\mathbb{I})\nabla u + F).$$

Using boundedness of $\mathcal{R}^{\mathbb{I}}_{1/2}$ from $T^2_{\beta+1/2}$ to $T^2_{\beta+1}$ for $\beta > -1$, we obtain

$$\|\mathcal{R}^A_{1/2}(F)\|_{T^2_{\beta+1}} \lesssim \|\nabla u\|_{T^2_{\beta+1/2}} + \|F\|_{T^2_{\beta+1/2}} \lesssim \|F\|_{T^2_{\beta+1/2}}.$$

The last inequality comes from (7.22) and (7.24). Thus, by density, $\mathcal{R}^A_{1/2}$ extends to a bounded operator from $T^2_{\beta+1/2}$ to $T^2_{\beta+1}$ for $-1/2-\epsilon < \beta < -1/2+\epsilon$.

It only remains to prove the claim. To this end, we consider the bounded map $\widetilde{\Phi}_A$ induced by Φ_A on the quotient space via the canonical projection modulo constants, noting that $\Phi_A(c) = (0,c)$ for any constant function c. We have the commutative diagram as shown in Figure 7.2.

$$\begin{tikzcd} S^2_{\beta+1/2} \arrow[r, "\Phi_A"] \arrow[d] & \operatorname{div} T^2_{\beta+1/2} \times (\dot{H}^{2\beta+1,2} + \mathbb{C}) \arrow[d] \\ S^2_{\beta+1/2}/\mathbb{C} \arrow[r, "\widetilde{\Phi}_A"] & \operatorname{div} T^2_{\beta+1/2} \times \dot{H}^{2\beta+1,2}/\mathbb{C} \end{tikzcd}$$

Figure 7.2: Quotient map of Φ_A

Proposition 7.25 implies $S^2_{\beta+1/2}/\mathbb{C}$ and $\operatorname{div} T^2_{\beta+1/2} \times \dot{H}^{2\beta+1,2}/\mathbb{C}$ are two isomorphic complex interpolation scales of Banach spaces. We also infer from the first point that $\widetilde{\Phi}_A$ is an isomorphism of Banach spaces for $\beta = -1/2$. Therefore, Sneiberg's Lemma (see [Šne74] or [KM98, Theorem 2.7]) yields there is a constant $\epsilon > 0$ so that for $-1/2-\epsilon < \beta < -1/2+\epsilon$, $\widetilde{\Phi}_A$ is an isomorphism of Banach spaces. By lifting, Φ_A is an isomorphism of semi-normed spaces. This completes the proof. □

7.3.3 Proof of Theorem 7.15

Let us first prove Lemma 7.22.

Proof of Lemma 7.22. Define the operator $\mathcal{R}^A_0$ on $L^2(\mathbb{R}^{1+n}_+)$ by

$$\mathcal{R}^A_0(F) := \nabla\mathcal{R}^A_{1/2}(F), \quad F \in L^2(\mathbb{R}^{1+n}_+). \tag{7.25}$$

It has been shown in [AP25, Theorem 3.1] that $\mathcal{R}_0^A$ extends to a bounded operator on T_0^∞. Let $F \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$ and $\widetilde{F} := (A-\mathbb{I})\mathcal{R}_0^A(F) + F$. Proposition 7.8 (2) says $\mathcal{R}_{1/2}^A(F) = \mathcal{R}_{1/2}^{\mathbb{I}}(\widetilde{F})$. So we get

$$\begin{aligned} \|\mathcal{R}_{1/2}^A(F)\|_{T^\infty_{1/2}} &= \|\mathcal{R}_{1/2}^{\mathbb{I}}(\widetilde{F})\|_{T^\infty_{1/2}} \lesssim \|\widetilde{F}\|_{T^\infty_0} \\ &\lesssim \|\mathcal{R}_0^A(F)\|_{T^\infty_0} + \|F\|_{T^\infty_0} \lesssim \|F\|_{T^\infty_0}, \end{aligned}$$

where the first inequality comes from boundedness of $\mathcal{R}_{1/2}^{\mathbb{I}}$ from T_0^∞ to $T_{1/2}^\infty$, see Theorem 6.22. It hence implies $\mathcal{R}_{1/2}^A$ extends to a bounded operator from T_0^∞ to $T_{1/2}^\infty$ by weak*-density. □

Now, we present the proof of Theorem 7.15.

Proof of Theorem 7.15. Let $\epsilon > 0$ be the constant given by Lemma 7.21. Define

$$\boxed{\beta_A := -1/2 - \epsilon/2}. \tag{7.26}$$

Let $\beta > \beta_A$ and $p^-_{\beta_A}(\beta) < p \le p^+_{\beta_A}(\beta)$.

The extension of $\mathcal{R}_{1/2}^A$ follows by interpolation of tent spaces, thanks to Lemmas 7.20, 7.21, and 7.22. The numbers $p^-_{\beta_A}(\beta)$ and $p^+_{\beta_A}(\beta)$ are exactly designed for this interpolation argument.

Let us verify the properties. First consider (a). We argue by density. For $F \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$, Proposition 7.8 (iii) says $u := \mathcal{R}_{1/2}^A(F)$ is a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F$. Then Corollary 7.4 gives

$$\|\nabla u\|_{T^p_{\beta+1/2}} \lesssim \|u\|_{T^p_{\beta+1}} + \|F\|_{T^p_{\beta+1/2}} \lesssim \|F\|_{T^p_{\beta+1/2}}.$$

So the gradient estimate holds for $F \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$. One can extend it to all $F \in T^p_{\beta+1/2}$ by density or weak*-density if $p = \infty$ (we shall omit to mention this subtlety in the sequel).

For (b), as $u \in T^p_{\beta+1}$ and $\nabla u \in T^p_{\beta+1/2}$, we get $u \in L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$. For any $F \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$, u satisfies $\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div} F$ in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$. Using the *a priori* estimates in (a), by density, one can extend the identity to all $F \in T^p_{\beta+1/2}$, valued in $\mathscr{D}'(\mathbb{R}^{1+n}_+)$. This proves (b).

To prove (c), we also argue by density. Proposition 7.8 (2) shows the formula (7.13) holds for $F \in T^p_{\beta+1/2} \cap L^2(\mathbb{R}^{1+n}_+)$. Note that all the operators involved have bounded extensions to $T^p_{\beta+1/2}$: For $\mathcal{R}_{1/2}^A$, $\nabla\mathcal{R}_{1/2}^A$, and $\mathcal{R}_{1/2}^{\mathbb{I}}$, it follows from (a); For $\mathcal{L}_1^{\mathbb{I}}$, it follows from Theorem 7.11 (a), as $\beta + 1/2 > -1/2$ and $p^-_{\beta_A}(\beta) > \frac{n}{n+2\beta+2} = p_{\mathbb{I}}(\beta+1/2)$. Then by density, it holds for all $F \in T^p_{\beta+1/2}$, valued in $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$.

Finally consider (d). Write $\widetilde{F} := (A-\mathbb{I})\nabla u + F$, so $u = \mathcal{R}_{1/2}^{\mathbb{I}}(\widetilde{F})$ (by (c)). As $\beta > \beta_A \ge -1$ and $p > p^-_{\beta_A}(\beta) \ge \frac{n}{n+2\beta+2}$, Theorem 6.22 (d) yields $\mathcal{R}_{1/2}^{\mathbb{I}}(\widetilde{F})$ lies in $C([0,\infty); \mathscr{S}')$, and so does u.

The proof of trace spaces is split into four cases.

Case 1: $p_{\beta_A}^-(\beta) < p \leq 2$ It follows from the same arguments right above, also using the formula in (c) and Theorem 6.22 (d).

Case 2: $2 < p < \infty$ We need to prove the trace in E_δ^q and $E_\delta^{-1,q}$ for any $\delta > 0$ and $q \in [p, \infty]$. The trace in $E_\delta^{-1,q}$ follows from the same arguments as in Case 1. In fact, it also works for $p = \infty$.

Let $\beta \geq -1/2$. To prove the trace in E_δ^q, by interpolation, it suffices to prove the trace in E_δ^p and E_δ^∞. To this end, we first observe that for any ball $B \subset \mathbb{R}^n$, Caccippoli's inequality (Lemma 7.3) yields

$$\begin{aligned} \|u(t)\|_{L^2(B)}^2 \lesssim t^{2\beta+1} \left(\frac{t}{r(B)^2} + 1 \right) \int_{t/2}^t \int_{2B} |s^{-(\beta+1)} u(s,y)|^2 ds dy \\ + t^{2\beta+1} \int_{t/2}^t \int_{2B} |s^{-(\beta+1/2)} F(s,y)|^2 ds dy, \end{aligned} \tag{7.27}$$

where the implicit constant does not depend on the center of B, nor t.

Let us introduce the *Carleson functional* defined by

$$\mathcal{C}(u)(x) := \sup_{B: x \in B} \left(\int_0^{r(B)^2} \fint_B |u(t,y)|^2 dt dy \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

where B describes balls in $\mathbb{R}^n$. For $0 < t < r(B)^2$, we infer from (7.27) the following pointwise estimate

$$\begin{aligned} \|u(t)\|_{L^2(B)} \lesssim |B|^{1/2} \ t^{\beta+1/2} \inf_{z \in B} \mathcal{C} \left(\mathbb{1}_{(t/2,t)}(s) s^{-(\beta+1)} u(s) \right)(z) \\ + |B|^{1/2} \ t^{\beta+1/2} \inf_{z \in B} \mathcal{C} \left(\mathbb{1}_{(t/2,t)}(s) s^{-(\beta+1/2)} F(s) \right)(z). \end{aligned} \tag{7.28}$$

Let $\delta > 0$ and $0 < t < \delta$. We first prove the trace in E_δ^p. For any $x \in \mathbb{R}^n$, applying (7.28) to the ball $B = B(x, \delta^{1/2})$ gives

$$\begin{aligned} \|u(t)\|_{E_\delta^p}^p \lesssim_\delta t^{(\beta+1/2)p} \int_{\mathbb{R}^n} dx \fint_{B(x,\delta^{1/2})} \left| \mathcal{C} \left(\mathbb{1}_{(t/2,t)}(s) s^{-(\beta+1)} u(s) \right)(z) \right|^p dz \\ + t^{(\beta+1/2)p} \int_{\mathbb{R}^n} dx \fint_{B(x,\delta^{1/2})} \left| \mathcal{C} \left(\mathbb{1}_{(t/2,t)}(s) s^{-(\beta+1/2)} F(s) \right)(z) \right|^p dz. \end{aligned}$$

Note that the A-C equivalence (see Proposition 3.11 or [CMS85, Theorem 3(b)]) says for $2 < p < \infty$,

$$\|f\|_{T_0^p} \eqsim \|\mathcal{C}(f)\|_{L^p(\mathbb{R}^n)}. \tag{7.29}$$

Therefore, we obtain

$$\|u(t)\|_{E^p_\delta} \lesssim_\delta t^{\beta+1/2} \left(\|\mathbb{1}_{(t/2,t)} u\|_{T^p_{\beta+1}} + \|\mathbb{1}_{(t/2,t)} F\|_{T^p_{\beta+1}} \right),$$

which tends to 0 as $t \to 0$ by Lebesgue's dominated convergence theorem. This proves the trace in E^p_δ.

Next, consider the trace in E^∞_δ. For any ball $B \subset \mathbb{R}^n$ with radius $\delta^{1/2}$, using (7.28) and (7.29) again, we obtain

$$\begin{aligned}
\|u(t)\|_{L^2(B)} \lesssim_\delta\ & t^{\beta+1/2} \left(\fint_B \left| \mathcal{C} \left(\mathbb{1}_{(t/2,t)}(s) s^{-(\beta+1)} u(s) \right)(x) \right|^p dx \right)^{1/p} \\
& + t^{\beta+1/2} \left(\fint_B \left| \mathcal{C} \left(\mathbb{1}_{(t/2,t)}(s) s^{-(\beta+1/2)} F(s) \right)(x) \right|^p dx \right)^{1/p} \\
\lesssim_\delta\ & t^{\beta+1/2} (\|\mathbb{1}_{(t/2,t)} u\|_{T^p_{\beta+1}} + \|\mathbb{1}_{(t/2,t)} F\|_{T^p_{\beta+1}}),
\end{aligned}$$

which tends to 0 as $t \to 0$. This concludes the case.

Case 3: $p = \infty$ This case is only concerned when $\beta \geq -1/2$. As aforementioned, it suffices to prove the trace in E^∞_δ for $\beta > -1/2$ and that in L^2_{loc} for $\beta = -1/2$. For $\beta > -1/2$, (7.27) implies for $0 < t < \delta$,

$$\|u(t)\|_{E^\infty_\delta} \lesssim_\delta t^{\beta+1/2} \left(\|u\|_{T^\infty_{\beta+1}} + \|F\|_{T^\infty_{\beta+1/2}} \right),$$

which tends to 0 as $t \to 0$. This proves the trace in E^∞_δ.

For $\beta = -1/2$, to prove the trace in L^2_{loc}, we fix a ball $B \subset \mathbb{R}^n$. For $0 < t < r(B)^2$, we infer from (7.27) that

$$\begin{aligned}
\sum_{j=0}^{+\infty} \|u(t/2^j)\|^2_{L^2(B)} &\lesssim \int_0^t \int_{2B} |s^{-1/2} u(s,y)|^2 ds dy + \int_0^t \int_{2B} |F(s,y)|^2 ds dy \\
&\lesssim_B \|u\|^2_{T^\infty_{1/2}} + \|F\|^2_{T^\infty_0}.
\end{aligned}$$

Since the series converges, we get $\|u(t/2^j)\|_{L^2(B)}$ tends to 0 as $j \to +\infty$. This implies $\|u(t)\|_{L^2(B)}$ tends to 0 as $t \to 0$ for any ball $B \subset \mathbb{R}^n$.

The proof is complete. □

7.3.4 Proof of Theorem 7.11 (c) and (d)

Let $\beta > -1/2$ and $p_A(\beta) < p \leq \infty$. Let $f \in T^p_\beta$ and $v = \mathcal{L}^A_1(f)$.

For (c), we first prove the identity for $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$. As $\nabla v \in T^p_{\beta+1/2}$ (by (a)), we infer from Theorems 6.11 and 6.22 that there is a unique global weak solution to the Cauchy problem

$$\begin{cases} \partial_t w - \Delta w = f + \operatorname{div}((A - \mathbb{I}) \nabla v) \\ w(0) = 0 \end{cases} \tag{7.30}$$

with $\nabla w \in T^p_{\beta+1/2}$, which is given by $w = \mathcal{L}^{\mathbb{I}}_1(f) + \mathcal{R}^{\mathbb{I}}_{1/2}((A-\mathbb{I})\nabla v)$. Meanwhile, as $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$, we get from Proposition 7.8 (i) that v itself is a weak solution to (7.30) with $\nabla v \in T^p_{\beta+1/2}$. Thus, by uniqueness, we obtain

$$v = \mathcal{L}^{\mathbb{I}}_1(f) + \mathcal{R}^{\mathbb{I}}_{1/2}((A-\mathbb{I})\nabla v)$$

as desired in (7.9) for any $f \in L^2_{\mathsf{c}}(\mathbb{R}^{1+n}_+)$.

Theorem 7.11 (a) and Theorem 7.15 (a) yield all the operators involved have bounded extension to T^p_β, so the density argument extends (7.9) to all $f \in T^p_\beta$ (or weak*-density if $p = \infty$). This proves (c).

To prove (d), we use the formula in (c). We infer from Theorem 6.11 (e) that $\mathcal{L}^{\mathbb{I}}_1(f)$ belongs to $C([0,\infty); \mathscr{S}')$ with traces in the spaces desired in (7.10).

Moreover, as $\beta > -1/2$ and $\nabla v \in T^p_{\beta+1/2}$, Theorem 7.15 (d) yields $\mathcal{R}^{\mathbb{I}}_{1/2}((A-\mathbb{I})\nabla v)$ belongs to $C([0,\infty); \mathscr{S}')$ with traces also in the spaces desired in (7.10).

Using the identity (7.9), we obtain $v \in C([0,\infty); \mathscr{S}')$ with traces in the spaces in (7.10). This proves (d). The proof is complete.

7.4 Homogeneous Cauchy problem

Consider the *homogeneous Cauchy problem*

$$\begin{cases} \partial_t w - \operatorname{div}(A\nabla w) = 0 \\ w(0) = w_0 \end{cases} . \tag{HC}$$

The main theorem of this section establishes extension of the propagator solution map $\mathcal{E}_A$ defined in (7.4) to $\dot{H}^{s,p}$ and hence proves the existence of weak solutions to (HC).

Theorem 7.26 (Extension of $\mathcal{E}_A$)**.** *Let $\beta_A < \beta < 0$ and $p^-_{\beta_A}(\beta) < p \le p^+_{\beta_A}(\beta)$, where β_A is given by Theorem 7.15. Then $\mathcal{E}_A$ extends to an operator from $\dot{H}^{2\beta+1,p}$ to $L^2_{\mathsf{loc}}((0,\infty); W^{1,2}_{\mathsf{loc}})$, also denoted by $\mathcal{E}_A$. The following properties hold for any $w_0 \in \dot{H}^{2\beta+1,p}$ and $w := \mathcal{E}_A(w_0)$.*

(a) (Regularity) ∇w *belongs to $T^p_{\beta+1/2}$ with the equivalence*

$$\|\nabla w\|_{T^p_{\beta+1/2}} \eqsim \|w_0\|_{\dot{H}^{2\beta+1,p}}. \tag{7.31}$$

Moreover, if $\beta_A < \beta < -1/2$, then w belongs to $T^p_{\beta+1}$ with the equivalence

$$\|w\|_{T^p_{\beta+1}} \eqsim \|\nabla w\|_{T^p_{\beta+1/2}} \eqsim \|w_0\|_{\dot{H}^{2\beta+1,p}}. \tag{7.32}$$

(b) (Explicit formula) *It holds that*

$$\begin{aligned} w &= \mathcal{E}_{\mathbb{I}}(w_0) + \mathcal{R}^A_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_{\mathbb{I}}(w_0)) \qquad (7.33) \\ &= \mathcal{E}_{\mathbb{I}}(w_0) + \mathcal{R}^{\mathbb{I}}_{1/2}((A-\mathbb{I})\nabla w). \qquad (7.34) \end{aligned}$$

(c) (Continuity) *w belongs to $C([0,\infty);\mathscr{S}')$ with $w(0)=w_0$.*

(d) (Strong continuity) *For $\beta=-1/2$ and $p_-(A)<p<p_+(A)$, w also belongs to $C_0([0,\infty);L^p)$ with*

$$\sup_{t\geq 0}\|w(t)\|_{L^p}\eqsim\|w_0\|_{L^p}.$$

(e) *w is a global weak solution to (HC) with initial data w_0.*

Remark 7.27. The equivalence (7.32) fails for $\beta\geq -1/2$.

For $\beta>-1/2$, as we shall see in Theorem 7.32, any weak solution $w\in T^p_{\beta+1}$ to the equation $\partial_t w-\operatorname{div}(A\nabla w)=0$ must be null for $p_A(\beta)<p\leq\infty$, even without imposing any initial condition.

For $\beta=-1/2$, it also fails. Instead, the equivalence holds in a larger class, called the *Kenig–Pipher space* X^p introduced by [KP93], which contains $T^p_{1/2}$. It has been shown in [AMP19, Corollaries 5.5, 5.10, and 7.5] and [Zat20, Theorem 7.6] that the following equivalence holds

$$\begin{cases}\|w\|_{X^p}\eqsim\|\nabla w\|_{T^p_0}\eqsim\|w_0\|_{L^p} & \text{if } p_-(A)=p^-_{\beta_A}(-1/2)<p<\infty\\ \|w\|_{X^\infty}\eqsim\|w_0\|_{L^\infty}, & \text{if } p=\infty\\ \|\nabla w\|_{T^\infty_0}\eqsim\|w_0\|_{\mathrm{BMO}} & \text{if } p=\infty\end{cases}.$$

The last equivalence corresponds to a special case $\beta=-1/2$ and $p=\infty$ in (7.31).

Remark 7.28. Let $\beta>-1$ and $\frac{n}{n+2\beta+2}<p\leq\infty$. If one can show that $\mathcal{R}^A_{1/2}$ extends to a bounded operator from $T^p_{\beta+1/2}$ to $T^p_{\beta+1}$, then the proof of Theorem 7.26 also works for initial data w_0 in $\dot{H}^{2\beta+1,p}$.

To prove this theorem, we first establish the notion of distributional traces for weak solutions to the equation $\partial_t u-\operatorname{div}(A\nabla u)=0$.

Proposition 7.29 (Trace). *Let $\beta>-1$ and $\frac{n}{n+2\beta+2}<p\leq\infty$. Let u be a global weak solution to $\partial_t u-\operatorname{div}(A\nabla u)=0$ with $\nabla u\in T^p_{\beta+1/2}$. Then there exists a unique $u_0\in\mathscr{S}'$ so that $u(t)$ converges to u_0 in $\mathscr{S}'$ as $t\to 0$, and moreover,*

$$u=\mathcal{E}_{\mathbb{I}}(u_0)+\mathcal{R}^{\mathbb{I}}_{1/2}((A-\mathbb{I})\nabla u). \tag{7.35}$$

In addition,

(i) *If $\beta\geq 0$ and $\frac{n}{n+2\beta}\leq p\leq\infty$, then u_0 is a constant.*

(ii) *If $-1<\beta<0$, then there exist $g\in\dot{H}^{2\beta+1,p}$ and $c\in\mathbb{C}$ so that $u_0=g+c$ with*

$$\|g\|_{\dot{H}^{2\beta+1,p}}\lesssim\|\nabla u\|_{T^p_{\beta+1/2}}.$$

Proof. The proof follows from a verbatim adaptation of Proposition 6.38 for time-independent coefficient matrices. □

Let us present the proof of Theorem 7.26.

Proof of Theorem 7.26. Let $\beta_A < \beta < 0$ and $p^-_{\beta_A}(\beta) < p \le p^+_{\beta_A}(\beta)$. Define the extension of $\mathcal{E}_A$ from $\dot{H}^{2\beta+1,p}$ to $L^2_{\mathsf{loc}}((0,\infty);W^{1,2}_{\mathsf{loc}})$ (verified below) by

$$\mathcal{E}_A(w_0) := \mathcal{E}_{\mathbb{I}}(w_0) + \mathcal{R}^A_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_{\mathbb{I}}(w_0)), \quad w_0 \in \dot{H}^{2\beta+1,p}. \tag{7.36}$$

This agrees with the formula in Proposition 7.8 (3) when $w_0 \in L^2$.

Let us verify that for any $w_0 \in \dot{H}^{2\beta+1,p}$, $\mathcal{E}_A(w_0)$ lies in $L^2_{\mathsf{loc}}((0,\infty);W^{1,2}_{\mathsf{loc}})$. Note that

$$\mathcal{E}_{\mathbb{I}}(w_0)(t,x) = (e^{t\Delta}w_0)(x), \quad (t,x) \in \mathbb{R}^{1+n}_+. \tag{7.37}$$

As $\dot{H}^{2\beta+1,p}$ embeds into $\mathscr{S}'$, we get that $\mathcal{E}_{\mathbb{I}}(w_0)$ lies in $C^\infty(\mathbb{R}^{1+n}_+)$, hence in $L^2_{\mathsf{loc}}((0,\infty);W^{1,2}_{\mathsf{loc}})$. Also, as $2\beta+1<1$, we invoke Theorem 5.17 (i) to get that $\nabla\mathcal{E}_{\mathbb{I}}(w_0)$ lies in $T^p_{\beta+1/2}$ with

$$\|\nabla\mathcal{E}_{\mathbb{I}}(w_0)\|_{T^p_{\beta+1/2}} \eqsim \|w_0\|_{\dot{H}^{2\beta+1,p}}. \tag{7.38}$$

Then applying Theorem 7.15 (a) gives

$$\|\mathcal{R}^A_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_{\mathbb{I}}(w_0))\|_{T^p_{\beta+1}} \lesssim \|\nabla\mathcal{E}_{\mathbb{I}}(w_0)\|_{T^p_{\beta+1/2}} \eqsim \|w_0\|_{\dot{H}^{2\beta+1,p}},$$

and

$$\|\nabla\mathcal{R}^A_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_{\mathbb{I}}(w_0))\|_{T^p_{\beta+1/2}} \lesssim \|w_0\|_{\dot{H}^{2\beta+1,p}}. \tag{7.39}$$

Thus, $\mathcal{R}^A_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_{\mathbb{I}}(w_0))$ lies in $L^2_{\mathsf{loc}}((0,\infty);W^{1,2}_{\mathsf{loc}})$ and so does $\mathcal{E}_A(w_0)$.

Write $w := \mathcal{E}_A(w_0)$. Let us prove the properties announced. First consider (a). The inequality $\|\nabla w\|_{T^p_{\beta+1/2}} \lesssim \|w_0\|_{\dot{H}^{2\beta+1,p}}$ directly follows from (7.38) and (7.39). The proof of the reverse inequality $\|w_0\|_{\dot{H}^{2\beta+1,p}} \lesssim \|\nabla w\|_{T^p_{\beta+1/2}}$ and (7.32) is deferred to the end of the proof.

Next, consider (b). The first formula (7.33) is exactly the definition of the extension (7.36). For the second (7.34), Proposition 7.8 (3) says it holds for $w_0 \in \dot{H}^{2\beta+1,p} \cap L^2$. Ensured by the *a priori* estimates in (a), one can extend the identity (valued in $L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$) to all $w_0 \in \dot{H}^{2\beta+1,p}$ by density or weak*-density if $p=\infty$.

For (c), we show that the two terms in (7.36) lie in $C([0,\infty);\mathscr{S}')$. Write

$$v := \mathcal{E}_{\mathbb{I}}(w_0), \quad u := \mathcal{R}^A_{1/2}((A-\mathbb{I})\nabla v).$$

As $p^-_{\beta_A}(\beta) < p \le p^+_{\beta_A}(\beta)$ and $\nabla v \in T^p_{\beta+1/2}$, Theorem 7.15 (d) yields $u \in C([0,\infty);\mathscr{S}')$ with $u(0)=0$. Moreover, by (7.37), we get that v lies in $C([0,\infty);\mathscr{S}')$ with $v(0)=w_0$, and so does $w=u+v$.

The statement (d) is extracted from [AMP19, Corollary 5.10 and Proposition 5.11].

To prove (e), notice that v is a global weak solution to the heat equation, and Theorem 7.15 (b) says u is a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = \operatorname{div}((A-\mathbb{I})\nabla v)$. Thus, $w = u+v$ is a global weak solution to $\partial_t w - \operatorname{div}(A\nabla w) = 0$. Moreover, (c) yields that $w(t)$ converges to w_0 as $t \to 0$ in $\mathscr{S}'$, hence in $\mathscr{D}'$. This proves (e).

Let us finish by proving the rest of (a). First, we show the reverse inequality $\|w_0\|_{\dot{H}^{2\beta+1,p}} \lesssim \|\nabla w\|_{T^p_{\beta+1/2}}$ in (7.31). Note that w is a global weak solution to $\partial_t w - \operatorname{div}(A\nabla w) = 0$ with $\nabla w \in T^p_{\beta+1/2}$, and $w(t)$ converges to w_0 in $\mathscr{S}'$ as $t \to 0$. Hence, w_0 agrees with the trace g of w constructed in Proposition 7.29, which gives $\|w_0\|_{\dot{H}^{2\beta+1,p}} \lesssim \|\nabla w\|_{T^p_{\beta+1/2}}$.

It only remains to prove (7.32). Using again the fact that w is a global weak solution to $\partial_t w - \operatorname{div}(A\nabla w) = 0$, we deduce from Corollary 7.4 that $\|\nabla w\|_{T^p_{\beta+1/2}} \lesssim \|w\|_{T^p_{\beta+1}}$.

On the other hand, Theorem 5.4 shows that when $\beta < -1/2$ (*i.e.*, $2\beta+1 < 0$), $\mathcal{E}_{\mathbb{I}}(w_0)$ belongs to $T^p_{\beta+1}$ with the equivalence

$$\|\mathcal{E}_{\mathbb{I}}(w_0)\|_{T^p_{\beta+1}} \eqsim \|w_0\|_{\dot{H}^{2\beta+1,p}}.$$

Then we infer from (7.36) that

$$\begin{aligned}\|w\|_{T^p_{\beta+1}} &\leq \|\mathcal{E}_{\mathbb{I}}(w_0)\|_{T^p_{\beta+1}} + \|\mathcal{R}^A_{1/2}((A-\mathbb{I})\nabla\mathcal{E}_{\mathbb{I}}(w_0))\|_{T^p_{\beta+1}} \\ &\lesssim \|w_0\|_{\dot{H}^{2\beta+1,p}} + \|\nabla\mathcal{E}_{\mathbb{I}}(w_0)\|_{T^p_{\beta+1/2}} \lesssim \|w_0\|_{\dot{H}^{2\beta+1,p}}.\end{aligned}$$

This proves (a). The proof is complete. □

Corollary 7.30 (Continuity of propagators)**.** *Let $p_-(A) < p < p_+(A)$ and $p_-(A)' < q < p_+(A)'$. Let $g \in L^p$ and $h \in L^q$.*

(i) *The function $t \mapsto \Gamma_A(t,s)g$ lies in $C_0([s,\infty); L^p)$.*

(ii) *The function $s \mapsto \Gamma_A(t,s)g$ lies in $C_{\mathsf{w}}([0,t]; L^p)$.* [2]

(iii) *The function $t \mapsto \Gamma_A(t,s)^*h$ lies in $C_{\mathsf{w}}([s,\infty); L^q)$.*

(iv) *The function $s \mapsto \Gamma_A(t,s)^*h$ lies in $C([0,t]; L^q)$.*

Proof. By duality, it suffices to prove (i) and (iv). The statement (i) follows from Theorem 7.26 (d) by shifting the time. We present two approaches to prove (iv). Both of them use the duality relation (7.2) in Corollary 7.7, which says

$$\Gamma_A(t,s)^* = \Gamma^-_{A^*}(s,t) = \Gamma_{\widetilde{A}_t}(t-s,0), \quad 0 \leq s \leq t,$$

[2]Here, $C_{\mathsf{w}}([0,t]; E)$ is the space of continuous functions valued in E equipped with its weak topology.

where $\widetilde{A}_t$ is defined in (7.3) given by

$$\widetilde{A}_t(s,x) := \begin{cases} A^*(t-s,x), & \text{if } 0 \le s \le t \\ \Lambda_0 \mathbb{I} & \text{if } s > t \end{cases}.$$

The first way is to adapt the proof of (i) to the backward equation. Detailed computation is left to the reader.

The second proof is based on the observation that

$$p_-(\widetilde{A}_t) \le p_+(A)' < p_-(A)' \le p_+(\widetilde{A}_t). \tag{7.40}$$

Indeed, since $(\Gamma_A(t,s))$ is uniformly bounded on L^p for $p_-(A) < p < p_+(A)$, by duality, $(\Gamma_A(t,s)^*)$ is uniformly bounded on L^q for $p_+(A)' < q < p_-(A)'$, and so is $(\Gamma_{\widetilde{A}_t}(\tau,0))_{0 \le \tau \le t}$. Moreover, as $p_-(\Lambda_0 \mathbb{I}) = 1$ and $p_+(\Lambda_0 \mathbb{I}) = \infty$, we get $(\Gamma_{\widetilde{A}_t}(\tau,0))_{\tau \ge 0}$ is uniformly bounded on L^q for $p_+(A)' < q < p_-(A)'$. Hence, (7.40) follows by shifting the time.

Theorem 7.26 (d) yields for $p_-(\widetilde{A}_t) < q < p_+(\widetilde{A}_t)$ (in particular for $p_+(A)' < q < p_-(A)'$) and $\psi \in L^q$, the function $\tau \mapsto \Gamma_{\widetilde{A}_t}(\tau,0)\psi$ belongs to $C_0([0,\infty); L^q)$. Applying it to the function $s \mapsto \Gamma_{\widetilde{A}_t}(t-s,0)h = \Gamma_A(t,s)^* h$ for $0 \le s \le t$ hence gives (iv). This completes the proof. □

7.5 Uniqueness and representation

Theorem 7.31 (Uniqueness)**.** *Let $\beta > \beta_A$ and $p^-_{\beta_A}(\beta) < p \le p^+_{\beta_A}(\beta)$. Let u be a global weak solution to the Cauchy problem*

$$\begin{cases} \partial_t u - \operatorname{div}(A \nabla u) = 0, \\ u(0) = 0 \end{cases},$$

with $\nabla u \in T^p_{\beta+1/2}$. Then $u = 0$.

We also present another class for which the uniqueness holds without imposing any initial condition.

Theorem 7.32. *Let $\beta > -1/2$ and $p_A(\beta) = p^-_{\beta_A}(\beta) < p \le p^+_{\beta_A}(\beta) = \infty$. Let $u \in T^p_{\beta+1}$ be a global weak solution to the equation $\partial_t u - \operatorname{div}(A\nabla u) = 0$. Then $u = 0$.*

The proofs are presented right below in Sections 7.5.1 and 7.5.2.

The following theorem is concerned with the representation of weak solutions in the solution class $\nabla u \in T^p_{\beta+1/2}$.

Theorem 7.33 (Representation)**.** *Let $\beta > \beta_A$ and $p^-_{\beta_A}(\beta) < p \le p^+_{\beta_A}(\beta)$. Let u be a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = 0$ with $\nabla u \in T^p_{\beta+1/2}$. Then u has a trace $u_0 \in \mathscr{S}'$, in the sense that $u(t)$ converges to u_0 in $\mathscr{S}'$ as $t \to 0$. Moreover,*

(i) *If* $\beta \geq 0$ *and* $\frac{n}{n+2\beta} \leq p \leq \infty$*, then* u *is a constant.*

(ii) *If* $\beta_A < \beta < 0$*, then there exist* $g \in \dot{H}^{2\beta+1,p}$ *and* $c \in \mathbb{C}$ *so that* $u_0 = g + c$ *and* $u = \mathcal{E}_A(g) + c$*, where* $\mathcal{E}_A$ *is the extension of the propagator solution map defined by* (7.4)*.*

The proof is deferred to Section 7.5.3.

7.5.1 Proof of Theorem 7.31

The following lemma reduces the proof to the case $p > p_-(A)$. Note that $p \leq p_-(A)$ occurs only if $\beta > -1/2$.

Lemma 7.34. *Let* $\beta > -1/2$ *and* $p_A(\beta) < p \leq p_-(A)$*. Then there exist* $\beta_0 > -1/2$ *and* $p_0 \in (p_-(A), 2)$ *so that* $T^p_{\beta+\gamma}$ *embeds into* $T^{p_0}_{\beta_0+\gamma}$ *for any* $\gamma \in \mathbb{R}$*.*

Proof. By embedding of tent spaces, it suffices to find $\beta_0 \in (-1/2, \beta)$ and $p_0 \in (p_-(A), 2)$ so that $2\beta_0 - \frac{n}{p_0} = 2\beta - \frac{n}{p}$. To do so, first pick β_1 with $2\beta_1 - \frac{n}{p_-(A)} = 2\beta - \frac{n}{p}$. As $p \leq p_-(A)$, we have $\beta_1 \leq \beta$. We claim $\beta_1 > -1/2$. If it holds, then perturbation gives $\beta_0 \in (-1/2, \beta_1) \subset (-1/2, \beta)$ and $p_0 \in (p_-(A), 2)$ with $2\beta_0 - \frac{n}{p_0} = 2\beta - \frac{n}{p}$.

It only remains to verify the claim. Note that

$$r > p_A(\gamma) = \frac{np_-(A)}{n + (2\gamma+1)p_-(A)} \iff 2\gamma - \frac{n}{r} > -1 - \frac{n}{p_-(A)}.$$

As $p > p_A(\beta)$, we get $2\beta_1 = 2\beta - \frac{n}{p} + \frac{n}{p_-(A)} > -1$, so $\beta_1 > -1/2$. $\square$

Let us present the proof of Theorem 7.31. Let u be a global weak solution to $\partial_t u - \operatorname{div}(A\nabla u) = 0$ with $\nabla u \in T^p_{\beta+1/2}$ and null initial data. The argument is split into 2 cases: $\beta \geq -1/2$ and $\beta < -1/2$.

7.5.1.1 $\beta \geq -1/2$

Thanks to Lemma 7.34, it suffices to consider the case $p_-(A) < p \leq \infty$. We infer from Lemma 5.22 that u satisfies the local L^2-norm estimate that for $0 < a < b < \infty$ and $R > 1$,

$$\int_a^b \int_{B(0,R)} |u|^2 \lesssim_{a,b,p,\beta} R^{3n+2} \left(\|\nabla u\|^2_{T^p_{\beta+1/2}} + \|u\|^2_{L^2((a,b)\times B(0,1))} \right).$$

In fact, it holds for any $u \in L^2_{\mathsf{loc}}(\mathbb{R}^{1+n}_+)$ with $\nabla u \in T^p_{\beta+1/2}$. This allows us to invoke [AMP19, Theorem 5.1] to get an interior representation of u, also called "*homotopy identity*", that for $0 < s < t < \infty$ and $h \in C^\infty_{\mathsf{c}}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} u(t,x)\overline{h}(x)dx = \int_{\mathbb{R}^n} u(s,x)\overline{\Gamma_A(t,s)^* h}(x)dx. \tag{7.41}$$

Meanwhile, as $p > p_A(\beta) > \frac{n}{n+2\beta+2}$ and $u(0) = 0$, Proposition 7.29 yields

$$u = \mathcal{R}^{\mathbb{I}}_{1/2}((A - \mathbb{I})\nabla u). \tag{7.42}$$

Hence, Theorem 7.15 (d) implies $u(s)$ tends to 0 as $s \to 0$ in $\mathscr{S}'$ and in some finer trace spaces depending on β and p.

Therefore, our strategy is to analyze the limit $s \to 0$ for (7.41), by employing these trace spaces and the continuity of the propagators, to obtain $u(t) = 0$ in $\mathscr{D}'$ for any $t > 0$, and hence $u = 0$. The argument is split into 3 sub-cases.

Case 1: $p_-(A) < p \le 2$ Using (7.42) and Theorem 7.15 (d), we get

$$\lim_{s\to 0} u(s) = 0 \quad \text{in } L^p.$$

Moreover, Corollary 7.30 (iv) says

$$\lim_{s\to 0} \Gamma_A(t,s)^* h = \Gamma_A(t,0)^* h \quad \text{in } L^{p'}.$$

Then taking the limit $s \to 0$ in (7.41) gives $u(t) = 0$ as desired.

Case 2: $2 < p < \infty$ Let $\delta > 0$. Using (7.42) and Theorem 7.15 (d) again, we get

$$\lim_{s\to 0} u(s) = 0 \quad \text{in } E^\infty_\delta.$$

For the other term, we invoke [AMP19, Lemma 4.7(2)] to get

$$\lim_{s\to 0} \Gamma_A(t,s)^* h = \Gamma_A(t,0)^* h \quad \text{in } E^1_\delta.$$

By duality, taking the limit $s \to 0$ in (7.41) gives $u(t) = 0$ as desired.

Case 3: $p = \infty$ For $\beta > -1/2$, it follows exactly the same arguments as in Case 2. To prove the case $\beta = -1/2$, we also use (7.41). Let $h \in C^\infty_{\mathsf{c}}(\mathbb{R}^n)$ and $M > 1$ so that $\operatorname{supp}(h) \subset B(0,M) =: B$. We claim that there exists a constant $c > 0$ so that for any $x \in \mathbb{R}^n$,

$$\sup_{0\le s\le t/2} \|\mathbb{1}_{B(x,1)} \Gamma_A(t,s)^* h\|_{L^2} \lesssim \|h\|_{L^2} \left(\mathbb{1}_{4B}(x) + e^{-c\frac{|x|^2}{t}} \mathbb{1}_{(4B)^c}(x) \right). \tag{7.43}$$

Indeed, when $x \in 4B$, using L^2-boundedness of $(\Gamma_A(t,s)^*)$, we have

$$\sup_{0\le s\le t/2} \|\mathbb{1}_{B(x,1)} \Gamma_A(t,s)^* h\|_{L^2} \le \sup_{0\le s\le t/2} \|\Gamma_A(t,s)^* h\|_{L^2} \le \|h\|_{L^2}.$$

When $x \notin 4B$, we get $\operatorname{dist}(B(x,1),B) \geq \frac{1}{2}|x|$. Using the $L^2 - L^2$ off-diagonal decay of $(\Gamma_A(t,s)^*)$, we obtain a constant $c > 0$ so that

$$\|\mathbb{1}_{B(x,1)}\Gamma_A(t,s)^*h\|_{L^2} \lesssim e^{-4c\frac{\operatorname{dist}(B(x,1),B)^2}{t-s}}\|h\|_{L^2} \leq e^{-c\frac{|x|^2}{t}}\|h\|_{L^2}$$

as desired. This proves (7.43).

Then taking averages over $B(x,1)$ for the integral on the right-hand side of (7.41), we obtain

$$\int_{\mathbb{R}^n} u(t,x)\overline{h}(x)dx = \int_{\mathbb{R}^n} dx \fint_{B(x,1)} u(s,y)\overline{\Gamma_A(t,s)^*h}(y)dy. \tag{7.44}$$

Remark 7.17 says $u(s)$ is uniformly bounded in E_1^∞ for $0 < s < 1$. Combining it with (7.43) gives that for $0 < s < \min\{t/2, 1\}$ and $x \in \mathbb{R}^n$,

$$\fint_{B(x,1)} |u(s)||\Gamma_A(t,s)^*h| \lesssim \|u\|_{E_1^\infty}\|h\|_{L^2}\left(\mathbb{1}_{4B}(x) + e^{-c\frac{|x|^2}{t}}\mathbb{1}_{(4B)^c}(x)\right),$$

which is integrable on $\mathbb{R}^n$. Moreover, Theorem 7.15 (d) says $u(s)$ tends to 0 as $s \to 0$ in L^2_{loc}, and Corollary 7.30 (iv) says $\Gamma_A(t,s)^*h$ converges to $\Gamma_A(t,0)^*h$ as $s \to 0$ in L^2. Therefore, we obtain that for any $x \in \mathbb{R}^n$,

$$\lim_{s\to 0}\fint_{B(x,1)} u(s,y)\overline{\Gamma_A(t,s)^*h}(y)dy = 0.$$

Applying Lebesgue's dominated convergence theorem to the limit $s \to 0$ of the integral on the right-hand side of (7.44) implies $u(t) = 0$.

This concludes the case $\beta \geq -1/2$.

7.5.1.2 $\beta < -1/2$

In this case, we use the interpolation argument as in Section 7.3.2. Recall the map Φ_A defined in (7.20) given by

$$\begin{aligned}\Phi_A : S^p_{\beta+1/2} &\to \operatorname{div} T^p_{\beta+1/2} \times (\dot{H}^{2\beta+1,p} + \mathbb{C})\\ \Phi_A(u) &:= (\partial_t u - \operatorname{div}(A\nabla u), \quad \operatorname{Tr}(u)).\end{aligned}$$

Lemma 7.24 implies Φ_A is bounded for $-1 < \beta < 0$ and $1 < p < \infty$. Let $\epsilon > 0$ be the constant given by Lemma 7.21. It has been shown in the proof of Lemma 7.21 that Φ_A is an isomorphism for $-1/2 - \epsilon < \beta \leq -1/2$ and $p = 2$. Let us recall the construction of the inverse Ψ_A defined in (7.21). For any $g \in \operatorname{div} T^p_{\beta+1/2}$ and $v_0 \in \dot{H}^{2\beta+1,p} + \mathbb{C}$, $v =: \Psi_A(g, v_0)$ is the unique weak solution in $S^p_{\beta+1/2}$ to the Cauchy problem

$$\begin{cases}\partial_t v - \operatorname{div}(A\nabla v) = g, \\ v(0) = v_0\end{cases}. \tag{7.45}$$

Let $\beta = -1/2$ and $p_-(A) = p^-_{\beta_A}(-1/2) < p < p^+_{\beta_A}(-1/2) = \infty$. For any $g \in \operatorname{div} T^p_0$ and $v_0 \in \dot{H}^{0,p} + \mathbb{C}$, Theorems 7.15 and 7.26 yield there is a weak solution $v \in S^p_0$ to (7.45) with

$$\|v\|_{S^p_0} \lesssim \|g\|_{\operatorname{div} T^p_0} + \|v_0\|_{\dot{H}^{0,p}}.$$

The proof in Section 7.5.1.1 ensures the weak solution is unique in S^p_0. Therefore, Ψ_A is bounded for $\beta = -1/2$ and $p_-(A) < p < \infty$.

Recall that (7.26) says

$$\beta_A = -1/2 - \epsilon/2.$$

Then by interpolation, we get Ψ_A extends to a bounded map from $\operatorname{div} T^p_{\beta+1/2} \times \dot{H}^{2\beta+1,p} + \mathbb{C}$ to $S^p_{\beta+1/2}$ for $\beta_A < \beta \leq -1/2$ and $p^-_{\beta_A}(\beta) < p \leq p^+_{\beta_A}(\beta)$. [3] By density, one also gets Ψ_A is the inverse of Φ_A. In particular, for $g = 0$ and $v_0 = 0$, it implies that the unique weak solution $v \in S^p_{\beta+1/2}$ to the Cauchy problem

$$\begin{cases} \partial_t v - \operatorname{div}(A\nabla v) = 0, \\ v(0) = 0 \end{cases} \tag{7.46}$$

must be null. Observe that any global weak solution v to (7.46) with $\nabla v \in T^p_{\beta+1/2}$ belongs to $S^p_{\beta+1/2}$, using the equation. We hence obtain the uniqueness in the class $\nabla v \in T^p_{\beta+1/2}$. This completes the proof.

7.5.2 Proof of Theorem 7.32

Let $\beta > -1/2$ and $p_A(\beta) < p \leq \infty$. Let $u \in T^p_{\beta+1}$ be a global weak solution to the equation $\partial_t u - \operatorname{div}(A\nabla u) = 0$. Using Lemma 7.34 again, we may assume $p_-(A) < p \leq \infty$. Still, our strategy is to use the homotopy identity (7.41). Note that as $u \in T^p_{\beta+1}$, Lemmas 5.21 and 6.19 yield that for $0 < a < b < \infty$ and $R \geq 1$,

$$\int_a^b \int_{B(0,R)} |u|^2 \lesssim_{a,b} R^n \left(\|u\|^2_{T^p_{\beta+1}} + \|u\|^2_{L^2((a,b)\times B(0,b^{1/2}))} \right),$$

which verifies the conditions in [AMP19, Theorem 5.1] to obtain the homotopy identity (7.41).

It remains to take the limits $s \to 0$ in (7.41) to get $u(t) = 0$ in $\mathscr{D}'$ for any $t > 0$ and hence $u = 0$. To show this, we first observe that for any $s > 0$, $u(s)$ lies in $E^p_{s/16}$ with

$$\|u(s)\|_{E^p_{s/16}} \lesssim s^{\beta+1/2} \|u\|_{T^p_{\beta+1}}.$$

[3] In fact, since both spaces are semi-normed, one needs to pass to the quotient map (from $\operatorname{div} T^p_{\beta+1/2} \times \dot{H}^{2\beta+1,p}/\mathbb{C}$ to $S^p_{\beta+1/2}/\mathbb{C}$) to use interpolation, and then lifts up, as shown in Figure 7.2. We leave the detailed verification to the reader.

Indeed, it follows by applying Caccioppoli's inequality (cf. Lemma 7.3) to the average of $u(s)$ on $B(x, \frac{\sqrt{s}}{4})$ as

$$\|u(s)\|_{E^p_{s/16}} \lesssim \left(\int_{\mathbb{R}^n} \left(\frac{1}{s} \int_{s/2}^{s} \fint_{B(x,\frac{\sqrt{s}}{2})} |u|^2 \right)^{p/2} dx \right)^{1/p} \lesssim s^{\beta+1/2} \|u\|_{T^p_{\beta+1}}.$$

Then the argument is split into two cases.

Case 1: $p_-(A) < p \leq 2$ Hölder's inequality yields that $E^p_{s/16}$ embeds into L^p, so we get

$$\|u(s)\|_{L^p} \lesssim \|u(s)\|_{E^p_{s/16}} \lesssim s^{\beta+1/2} \|u\|_{T^p_{\beta+1}},$$

which tends to 0 as $s \to 0$. Meanwhile, Corollary 7.30 (iv) says

$$\lim_{s \to 0} \Gamma_A(t,s)^* h = \Gamma(t,0)^* h \quad \text{ in } L^{p'}.$$

Then taking the limit $s \to 0$ in (7.41) implies $u(t) = 0$ as desired.

Case 2: $2 < p \leq \infty$ Let $\delta > 0$ and $0 < s < t < \delta$. Using the change of aperture for slice spaces (cf. (3.2)), we get $E^p_{s/16}$ embeds into E^p_δ with

$$\|u(s)\|_{E^p_\delta} \lesssim \|u(s)\|_{E^p_{s/16}} \lesssim s^{\beta+1/2} \|u\|_{T^p_{\beta+1}},$$

which tends to 0 as $s \to 0$. On the other hand, [AMP19, Lemma 4.7(2)] says

$$\lim_{s \to 0} \Gamma_A(t,s)^* h = \Gamma_A(t,0)^* h \quad \text{ in } E^{p'}_\delta.$$

Taking the limit $s \to 0$ in (7.41) implies $u(t) = 0$.

This completes the proof.

7.5.3 Proof of Theorem 7.33

Let $\beta > \beta_A$ and $p^-_{\beta_A}(\beta) < p \leq p^+_{\beta_A}(\beta)$. Let u be a global weak solution to $\partial_t u - \operatorname{div}(A \nabla u) = 0$ with $\nabla u \in T^p_{\beta+1/2}$. As $p > p^-_{\beta_A}(\beta) > \frac{n}{n+2\beta+2}$, Proposition 7.29 yields there exists a unique $u_0 \in \mathscr{S}'$ so that $u(t)$ converges to u_0 as $t \to 0$, and

$$u = \mathcal{E}_{\mathbb{I}}(u_0) + \mathcal{R}^{\mathbb{I}}_{1/2}((A - \mathbb{I}) \nabla u).$$

Then we prove the properties of the trace. First consider (i). For $\beta \geq 0$ and $\frac{n}{n+2\beta} \leq p \leq \infty$, Proposition 7.29 (i) says u_0 is a constant, and so is $\mathcal{E}_{\mathbb{I}}(u_0)$. Thus, $v := u - \mathcal{E}_{\mathbb{I}}(u_0)$ is a global weak solution to the Cauchy problem

$$\begin{cases} \partial_t v - \operatorname{div}(A \nabla v) = 0 \\ v(0) = 0 \end{cases}$$

with $\nabla v \in T^p_{\beta+1/2}$. As $p^-_{\beta_A}(\beta) < p \leq p^+_{\beta_A}(\beta) = \infty$, Theorem 7.31 yields $v = 0$, so $u = \mathcal{E}_{\mathbb{I}}(u_0)$ is a constant. This proves (i).

Next, to prove (ii), note that Proposition 7.29 (ii) says there exist $g \in \dot{H}^{2\beta+1,p}$ and $c \in \mathbb{C}$ so that $u_0 = g + c$. Then we use the formula (7.34) proved in Theorem 7.26 (b) to get

$$u = \mathcal{E}_{\mathbb{I}}(g) + \mathcal{R}^{\mathbb{I}}_{1/2}((A - \mathbb{I})\nabla u) + c = \mathcal{E}_A(g) + c.$$

This proves (ii). The proof is complete.

Chapter 8

Navier–Stokes equations

“我醉欲眠卿且去，
明朝有意抱琴来。” [1]

《山中与幽人对酌》李白

This chapter is devoted to an application of the tent space theory to the Navier–Stokes equation, as an illustrating example for applications towards non-linear equations. It contains the article [Hou24] “*On regularity of solutions to the Navier–Stokes equation with initial data in* BMO^{-1}”.

8.1 Introduction

This chapter is concerned with regularity of mild solutions to the Cauchy problem of the incompressible Navier–Stokes equation

$$\begin{cases} \partial_t u + (u \cdot \nabla) u = \Delta u - \nabla p, \\ \nabla \cdot u = 0, \\ u(0) = u_0, \quad \nabla \cdot u_0 = 0, \end{cases} \tag{NS}$$

where $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ denotes the unknown velocity vector, and $p : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ denotes the unknown scalar pressure. Let $0 < T \leq \infty$. For a divergence-free tempered distribution u_0, we say u is a *mild* solution to (NS) with initial data u_0 if it satisfies the integral equation

$$u(t) = e^{t\Delta} u_0 - B(u,u)(t)$$

[1]English translation:
I'm drunk and ready for my sleep –
So, friend, take your leave and keep.
But should tomorrow your heart invite,
Come bearing your lute to share delight. Written by Li Bai.

in the sense of distributions $\mathscr{D}'((0,T)\times\mathbb{R}^n)$, [2] where the bilinear operator B is formally given by the integral

$$B(u,u)(t) := \int_0^t e^{(t-s)\Delta}\mathbb{P}\,\mathrm{div}(u\otimes u)(s)ds, \quad t\in(0,T). \tag{8.1}$$

Here, $\mathbb{P}$ denotes the Leray projection on divergence-free vector fields.

Based on the scaling property of the equation

$$u(t,x)\mapsto \lambda u(\lambda^2 t, \lambda x),$$

there is a chain of scale-invariant (or called *critical*) spaces

$$\dot{H}^{\frac{n}{2}-1}\hookrightarrow L^n \hookrightarrow \dot{B}^{-1+\frac{n}{p}}_{p,\infty}\hookrightarrow \mathrm{BMO}^{-1}\hookrightarrow \dot{B}^{-1}_{\infty,\infty}, \quad 2\le n<p<\infty.$$

The pioneering work of Fujita and Kato [FK64] (together with a later complement by Chemin [Che92]) establishes global existence of mild solutions $u\in C([0,\infty);\dot{H}^{n/2-1})$, provided that the initial data u_0 is small. [3] Such existence results are also valid in L^n due to Kato [Kat84] and in $\dot{B}^{-1+n/p}_{p,\infty}$ (with weak*-topology) due to Cannone [Can97] and Planchon [Pla98]. On the other hand, the work of Bourgain and Pavlović [BP08] addresses ill-posedness of (NS) in $\dot{B}^{-1}_{\infty,\infty}$. See also the works of Yoneda [Yon10] and Wang [Wan15].

For BMO^{-1}, Koch–Tataru [KT01] establish global existence of mild solutions $u\in X_\infty$ (see Section 8.2 for definition) for small initial data. Auscher–Dubois–Tchamitchian [ADT04] further show that the solution u belongs to $L^\infty((0,\infty);\mathrm{BMO}^{-1})$. Miura–Sawada [MS06] and Germain–Pavlović–Staffilani [GPS07] obtain spatial analyticity of u. But time regularity remains unknown.

The main goal of this chapter is to address this issue with the following result.

Theorem 8.1. *Let $0<T\le\infty$. Let $u_0\in\mathrm{BMO}^{-1}$ be divergence free and $u\in X_T$ be a mild solution to the Navier–Stokes equation (NS) with initial data u_0. Then u belongs to $C_{\mathsf{w}}([0,T);\mathrm{BMO}^{-1})$ with $u(0)=u_0$ and*

$$\sup_{0\le t<T}\|u(t)\|_{\mathrm{BMO}^{-1}}\lesssim \|u_0\|_{\mathrm{BMO}^{-1}}+\|u\|^2_{X_T}. \tag{8.2}$$

Here, $C_{\mathsf{w}}([0,T);\mathrm{BMO}^{-1})$ is the space of weakly* continuous functions valued in BMO^{-1}, where BMO^{-1} is endowed with the weak*-topology with respect to the homogeneous Hardy–Sobolev space $\dot{H}^{1,1}$, see Section 8.2 for details.

[2] We denote spaces of scalar-valued and spaces of vector-valued functions or distributions in the same way.

[3] Strictly speaking, Fujita and Kato's work concerns inhomogeneous Sobolev spaces defined by powers of the Stokes operator on bounded domains.

The next result is concerned with the long-time behavior of global mild solutions. We refer the reader to [GIP02] for motivation to study the asymptotic behavior of global mild solutions, and to [Pla98, GIP03] for results in other critical spaces.

Theorem 8.2. *Let $u_0 \in \mathrm{BMO}^{-1}$ be divergence free and $u \in X_\infty$ be a global mild solution to the Navier–Stokes equation* (NS) *with initial data u_0. Then $u(t)$ converges to 0 as $t \to \infty$ in* BMO^{-1} *(also endowed with the weak*-topology).*

Let us briefly discuss the earlier works. Koch–Tataru [KT01] establish local well-posedness of (NS) for initial data in the closure of Schwartz functions in BMO^{-1}, denoted by VMO^{-1}. More precisely, for any divergence-free $u_0 \in \mathrm{VMO}^{-1}$, there exist a time $T > 0$ and a mild solution $u \in X_T$ to (NS) with initial data u_0, which also satisfies

$$\lim_{\tau \to 0+} \|u\|_{X_\tau} = 0. \tag{8.3}$$

Dubois [Dub02] proves that any mild solution $u \in X_T$ with initial data $u_0 \in \mathrm{VMO}^{-1}$ is strongly continuous in time, valued in BMO^{-1}.

Next, Dubois [Dub02] and Auscher–Dubois–Tchamitchian [ADT04] present the long-time limit of u, namely, any global mild solution $u \in X_\infty$ with initial data $u_0 \in \mathrm{VMO}^{-1}$ satisfies

$$\lim_{t \to \infty} \sqrt{t}\|u(t)\|_{L^\infty} = 0, \quad \lim_{t \to \infty} \|u(t)\|_{\mathrm{BMO}^{-1}} = 0. \tag{8.4}$$

See also the work of Germain [Ger06] for a particular 2D case.

Therefore, Theorems 8.1 and 8.2 could be understood as the complement of [Dub02, ADT04] without local smallness (8.3).

Remark 8.3. In fact, Theorems 8.1 and 8.2 are sharp.

(i) The heat semigroup $(e^{t\Delta})$ is merely weak*-continuous on BMO^{-1}, so the weak*-continuity of mild solutions with BMO^{-1}-initial data, as asserted in Theorem 8.1, might possibly be the best expected. In comparison, $(e^{t\Delta})$ is strongly continuous in VMO^{-1}, which aligns with the strong continuity of mild solutions with VMO^{-1}-initial data, as obtained by Dubois.

(ii) The long-time behavior in Theorem 8.2 fails if we consider the strong topology of BMO^{-1}. Indeed, let $u_0 \in \mathrm{BMO}^{-1}$ be non-zero, divergence free, and homogeneous of degree -1, with small BMO^{-1}-norm. Then there exists a unique small self-similar global mild solution $u \in X_\infty$ to (NS) with initial data u_0, i.e.,

$$u(t,x) = \frac{1}{\sqrt{t}} U\left(\frac{x}{\sqrt{t}}\right), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n,$$

for some non-zero divergence-free $U \in C^\infty(\mathbb{R}^n)$, see [GPS07, Theorem 2.7]. In particular, by scale invariance, one gets that for such self-similar u,

$$\|u(t)\|_{\mathrm{BMO}^{-1}} = \|U\|_{\mathrm{BMO}^{-1}} \neq 0, \quad t > 0.$$

Therefore, it never vanishes in the strong topology.

This argument does not conflict with the results by Auscher, Dubois, and Tchamitchian for VMO^{-1}-initial data, cf. (8.4), as we notice that there is no non-zero distribution in VMO^{-1} that is homogeneous of degree -1.

To finish the introduction, let us mention the problem of uniqueness. To our best knowledge, uniqueness of mild solutions for arbitrary initial data in BMO^{-1} still remains open. Some partial results require additional smallness (e.g. [KT01]) or local smallness (e.g. [Dub02, Miu05]) of the solution. For instance, Dubois [Dub02] proves that any mild solution $u \in X_T$ satisfying (8.3) with $u_0 \in \mathrm{BMO}^{-1}$ (in particular for $u_0 \in \mathrm{VMO}^{-1}$) is unique. However, we also notice that a recent paper of Guillod–Šverák [GŠ23] provides numerical evidence suggesting non-uniqueness for large initial data in BMO^{-1} by symmetry breaking.

Organization

In Section 8.2, we introduce the function spaces and their basic properties to be used. Section 8.3 outlines the proofs of Theorems 8.1 and 8.2, with two main propositions (cf. Propositions 8.6 and 8.7) proved in Sections 8.4 and 8.5, respectively.

8.2 Function spaces

This section collects definitions and fundamental properties of function spaces to be used in this chapter. Most results are drawn from Chapter 3 and Section 5.1, while some notations may differ.

Let BMO^{-1} be the collection of distributions $f \in \mathscr{D}'(\mathbb{R}^n)$ with $f = \operatorname{div} g$ for some $g \in \mathrm{BMO}$. The space BMO^{-1} is isomorphic to the homogeneous Triebel–Lizorkin space $\dot{F}^{-1}_{\infty,2}$ and the homogeneous Hardy–Sobolev space $\dot{H}^{-1,\infty}$. In particular, let us mention two basic properties to be frequently used in this chapter.

- (Duality) The space BMO^{-1} identifies with the dual of $\dot{H}^{1,1}$ via $L^2(\mathbb{R}^n)$-duality.

- (Density) Let $\mathscr{S}_\infty$ be the subspace of Schwartz functions $\mathscr{S}$ consisting of $\phi \in \mathscr{S}$ with $\int_{\mathbb{R}^n} x^\alpha \phi(x) dx = 0$ for any multi-index α. It is dense in $\dot{H}^{s,p}$ for any $s \in \mathbb{R}$ and $0 < p < \infty$.

For $1 \le q < \infty$ and $0 < T \le \infty$, define the *Carleson functional* $\mathcal{C}_T^{(q)}$ on (vector-valued, strongly) measurable functions u on $(0,T) \times \mathbb{R}^n$ by

$$\mathcal{C}_T^{(q)}(u)(x) := \sup_{\substack{B: x \in B \\ r(B)^2 < T}} \left(\int_0^{r(B)^2} \fint_B |u(t,y)|^q dt dy \right)^{1/q}, \quad x \in \mathbb{R}^n,$$

where B describes balls in $\mathbb{R}^n$. Let X_T be the collection of measurable functions u on $(0,T) \times \mathbb{R}^n$ for which

$$\|u\|_{X_T} := \sup_{0<t<T} \|t^{1/2} u(t)\|_{L^\infty} + \|\mathcal{C}_T^{(2)}(u)\|_{L^\infty} < \infty.$$

To have a better understanding of the norm $\|\mathcal{C}_T^{(q)}(u)\|_{L^\infty}$, we recall tent spaces introduced by [CMS85]. Here we adapt to the parabolic settings. Let $\beta \in \mathbb{R}$ and $1 \le q < \infty$. The *tent space* $T_\beta^{\infty,q}$ consists of measurable functions u on $\mathbb{R}_+^{1+n}$ for which

$$\|u\|_{T_\beta^{\infty,q}} := \sup_B \left(\int_0^{r(B)^2} \fint_B |t^{-\beta} u(t,y)|^q dt dy \right)^{1/q} < \infty.$$

For coherence, we also write

$$\|u\|_{L_\beta^\infty(\mathbb{R}_+^{1+n})} := \sup_{t>0} \|t^{-\beta} u(t)\|_{L^\infty}.$$

Observe that

$$\|u\|_{T_\beta^{\infty,q}} = \|\mathcal{C}_\infty^{(q)}(t^{-\beta} u(t))\|_{L^\infty}.$$

In fact, most of the properties of tent spaces apply *mutatis mutandis* to the norm $\|\mathcal{C}_T^{(q)}(t^{-\beta} u(t))\|_{L^\infty}$ for $T < \infty$. We shall omit to mention this in the sequel.

For $\beta \in \mathbb{R}$ and $1 < q < \infty$, $T_\beta^{\infty,q}$ identifies with the dual of $T_{-\beta}^{1,q'}$ via $L^2(\mathbb{R}_+^{1+n})$-duality, where the *tent space* $T_{-\beta}^{1,q'}$ consists of measurable functions u on $\mathbb{R}_+^{1+n}$ for which

$$\|u\|_{T_{-\beta}^{1,q'}} := \int_{\mathbb{R}^n} \left(\int_0^\infty \fint_{B(x,t^{1/2})} |t^\beta u(t,y)|^{q'} dt dy \right)^{1/q'} dx < \infty.$$

For $q = 1$, the duality is more subtle (see [CMS85, Proposition 1]) but for any measurable functions u, v on $\mathbb{R}_+^{1+n}$, it still holds that

$$\int_0^\infty \int_{\mathbb{R}^n} |u(t,y)||v(t,y)| dt dy \lesssim \|u\|_{T_\beta^{\infty,1}} \|v\|_{T_{-\beta}^{1,\infty}}, \tag{8.5}$$

where the $T^{1,\infty}_{-\beta}$-norm is given by

$$\|v\|_{T^{1,\infty}_{-\beta}} := \int_{\mathbb{R}^n} \left(\operatorname*{esssup}_{t>0,|y-x|<t^{1/2}} |t^\beta v(t,y)| \right) dx.$$

A well-known heat characterization of BMO^{-1} provides the link between BMO^{-1} and the tent space $T^{\infty,2}$ as

$$\|u_0\|_{\mathrm{BMO}^{-1}} \eqsim \|e^{t\Delta}u_0\|_{T^{\infty,2}}.$$

Here and in the sequel, we omit the script β for tent spaces if $\beta = 0$.

8.3 Outline of the proofs

In this section, we outline the proofs of Theorems 8.1 and 8.2. It is enough to prove the case $T = \infty$. The same argument also works when T is finite. Let $u \in X_\infty$ be a (global) mild solution to (NS) with initial data $u_0 \in \mathrm{BMO}^{-1}$. Hölder's inequality yields

$$u \otimes u \in L^\infty_{-1}(\mathbb{R}^{1+n}_+) \cap T^{\infty,2}_{-1/2} \cap T^{\infty,1}.$$

Define the linear operator

$$\mathcal{L}(f)(t) := \int_0^t e^{(t-s)\Delta}\mathbb{P}\operatorname{div} f(s)ds, \quad t > 0.$$

Theorems 8.1 and 8.2 follow from the following main proposition.

Proposition 8.4. *Let $f \in L^\infty_{-1}(\mathbb{R}^{1+n}_+) \cap T^{\infty,2}_{-1/2} \cap T^{\infty,1}$. Then $\mathcal{L}(f)$ belongs to $C_{\mathsf{w},0}([0,\infty);\mathrm{BMO}^{-1})$ with $\mathcal{L}(f)(0) = 0$.*

Here, $C_{\mathsf{w},0}([0,\infty);\mathrm{BMO}^{-1})$ is the space of weakly* continuous functions valued in BMO^{-1} (endowed with the weak*-topology), with limit 0 as $t \to \infty$.

Admitting Proposition 8.4, let us prove Theorems 8.1 and 8.2.

Proof of Theorems 8.1 and 8.2, admitting Proposition 8.4. Note that the heat semigroup $(e^{t\Delta})$ is bounded and weak*-continuous on BMO^{-1}, one can find the function $t \mapsto e^{t\Delta}u_0$ lies in $C_{\mathsf{w},0}([0,\infty);\mathrm{BMO}^{-1})$ with

$$\sup_{t\geq 0} \|e^{t\Delta}u_0\|_{\mathrm{BMO}^{-1}} \eqsim \|u_0\|_{\mathrm{BMO}^{-1}}. \tag{8.6}$$

Moreover, Proposition 8.4 yields

$$B(u,u) = \mathcal{L}(u \otimes u)$$

also belongs to $C_{\mathsf{w},0}([0,\infty);\mathrm{BMO}^{-1})$ with $B(u,u)(0) = 0$. We hence obtain that u lies in $C_{\mathsf{w},0}([0,\infty);\mathrm{BMO}^{-1})$ with $u(0) = u_0$.

To show the bound (8.2), we infer from [ADT04, Lemma 8] that

$$\sup_{t>0} \|B(u,u)(t)\|_{\mathrm{BMO}^{-1}} \lesssim \|u\|_{X_\infty}^2. \tag{8.7}$$

In fact, this inequality extends to $t \geq 0$ due to the fact that $B(u,u)(t)$ tends to 0 in BMO^{-1} as $t \to 0+$. Combining this with (8.6) gives the bound (8.2). This completes the proof. □

Now we concentrate on the proof of Proposition 8.4. As we shall see, our proof also implies the bound (8.7), see Remark 8.11.

Proof of Proposition 8.4. We follow [AF17] to decompose $\mathcal{L}$ as

$$\begin{aligned}\mathcal{L}(f)(t) = &\int_0^t \Delta e^{(t-s)\Delta}(s\Delta)^{-1}(\mathbb{I}-e^{2s\Delta})s^{1/2}\mathbb{P}\operatorname{div} s^{1/2}f(s)ds \\ &+ \int_0^t e^{(t+s)\Delta}\mathbb{P}\operatorname{div} f(s)ds.\end{aligned}$$

As $f \in L^\infty_{-1}(\mathbb{R}^{1+n}_+) \cap T^{\infty,2}_{-1/2}$, we get $s^{1/2}f(s) \in L^\infty_{-1/2}(\mathbb{R}^{1+n}_+) \cap T^{\infty,2}$. Define the operator

$$\mathcal{R}(F)(s) := (s\Delta)^{-1}(\mathbb{I}-e^{2s\Delta})s^{1/2}\mathbb{P}\operatorname{div} F(s), \quad s>0.$$

Lemma 8.5. *$\mathcal{R}$ is bounded on both $L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)$ and $T^{\infty,2}$.*

We defer the proof to the end of this section. Now we have

$$g(s) := (s\Delta)^{-1}(\mathbb{I}-e^{2s\Delta})s^{1/2}\mathbb{P}\operatorname{div} s^{1/2}f(s)$$

belongs to $L^\infty_{-1/2}(\mathbb{R}^{1+n}_+) \cap T^{\infty,2}$. Define the *maximal regularity operator*

$$\mathcal{L}_0(g)(t) := \int_0^t \Delta e^{(t-s)\Delta}g(s)ds, \quad t>0,$$

and the remainder

$$\mathcal{L}_1(f)(t) := \int_0^t e^{(t+s)\Delta}\mathbb{P}\operatorname{div} f(s)ds, \quad t>0.$$

Proposition 8.4 directly follows from the following two propositions.

Proposition 8.6. *Let $g \in L^\infty_{-1/2}(\mathbb{R}^{1+n}_+) \cap T^{\infty,2}$. Then $\mathcal{L}_0(g)$ lies in $C_{\mathsf{w},0}([0,\infty);\mathrm{BMO}^{-1})$ with $\mathcal{L}_0(g)(0)=0$.*

Proposition 8.7. *Let $f \in T^{\infty,1}$. Then $\mathcal{L}_1(f)$ lies in $C_{\mathsf{w},0}([0,\infty);\mathrm{BMO}^{-1})$ with $\mathcal{L}_1(f)(0)=0$.*

The proofs are provided in Sections 8.4 and 8.5, respectively. □

To end this section, let us prove Lemma 8.5.

Proof of Lemma 8.5. For $s>0$, define the operator $\mathcal{R}_s$ on $L^2(\mathbb{R}^n)$ as

$$\mathcal{R}_s(h) := (s\Delta)^{-1}(\mathbb{I} - e^{2s\Delta})s^{1/2}\mathbb{P}\operatorname{div} h.$$

Using Fourier transform, one gets that the family $(\mathcal{R}_s)_{s>0}$ is uniformly bounded on $L^2(\mathbb{R}^n)$. Denote by $K_s(x)$ the convolution kernel of $\mathcal{R}_s$. We infer from singular integral realization of pseudo-differential operators (see e.g. [Ste93, §VI.4]) that the kernel satisfies the pointwise bound

$$|K_s(x)| \lesssim s^{-n/2}\min\{(s^{-1/2}|x|)^{-n+1}, (s^{-1/2}|x|)^{-n-1}\} \tag{8.8}$$

for any $s>0$ and $x \in \mathbb{R}^n \setminus \{0\}$. In particular, it lies in $L^1(\mathbb{R}^n)$ with a bound independent of s. Thus, $(\mathcal{R}_s)$ is uniformly bounded on $L^\infty(\mathbb{R}^n)$, and we hence get boundedness of $\mathcal{R}$ on $L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)$ as

$$\begin{aligned}\|\mathcal{R}(F)\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)} &= \sup_{s>0}\|s^{1/2}\mathcal{R}_s(F(s))\|_{L^\infty} \\ &\lesssim \sup_{s>0}\|s^{1/2}F(s)\|_{L^\infty} = \|F\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}.\end{aligned}$$

Next, boundedness of $\mathcal{R}$ on $T^{\infty,2}$ follows from [AF17, Lemma 3.1] with slight modifications of the proof, once we show that for any $s>0$, $E,F\subset\mathbb{R}^n$ as Borel sets with $\operatorname{dist}(E,F)\geq s^{1/2}$, and $h\in L^2$,

$$\|\mathbb{1}_F\mathcal{R}_s(\mathbb{1}_E h)\|_{L^\infty} \lesssim s^{-\frac{n}{4}}\left(s^{-1}\operatorname{dist}(E,F)^2\right)^{-\frac{1}{4}(n+1)}\|\mathbb{1}_E h\|_{L^2}. \tag{8.9}$$

To verify (8.9), we notice that the kernel bound (8.8) implies

$$\|\mathbb{1}_F\mathcal{R}_s(\mathbb{1}_E h)\|_{L^\infty} \lesssim s^{-\frac{n}{2}}\left(s^{-1}\operatorname{dist}(E,F)^2\right)^{-\frac{1}{2}(n+1)}\|\mathbb{1}_E h\|_{L^1}.$$

Since $(\mathcal{R}_s)$ is uniformly bounded on L^∞, we get (8.9) by interpolation. This completes the proof. □

In a nutshell, it only remains to prove Propositions 8.6 and 8.7. The proofs are presented in the following two sections.

8.4 Proof of Proposition 8.6

This section is concerned with the proof of Proposition 8.6. We first treat the continuity at $t=0$ and then at $t>0$. Finally we show the limit as $t\to\infty$. Recall that BMO^{-1} is equipped with the weak*-topology with respect to $\dot{H}^{1,1}$.

8.4.1 Continuity at $t = 0$

To prove $\mathcal{L}_0(g)(t)$ tends to 0 in BMO^{-1} as $t \to 0+$, we argue by duality. Pick $\varphi \in \mathscr{S}_\infty$. Fubini's theorem yields that for any $t > 0$,

$$\langle \mathcal{L}_0(g)(t), \varphi \rangle_{L^2(\mathbb{R}^n)} = \int_0^t \int_{\mathbb{R}^n} g(s,y) e^{(t-s)\Delta} \Delta\varphi(y) ds dy.$$

It suffices to verify that

$$\Phi := \int_0^t \int_{\mathbb{R}^n} |g(s,y)||e^{(t-s)\Delta} \Delta\varphi(y)| ds dy \xrightarrow[t \to 0+]{} 0. \tag{8.10}$$

To achieve this, we split Φ into two parts, the main term

$$\Phi_1 := \int_0^{t/2} \int_{\mathbb{R}^n} |g(s,y)||e^{(t-s)\Delta} \Delta\varphi(y)| ds dy,$$

and the remainder

$$R_1 := \int_{t/2}^t \int_{\mathbb{R}^n} |g(s,y)||e^{(t-s)\Delta} \Delta\varphi(y)| ds dy.$$

8.4.1.1 Estimates of Φ_1

Thanks to duality of tent spaces (see [CMS85, Theorem 1(a)]), we have

$$\Phi_1 \lesssim \|\mathcal{C}^{(2)}_{t/2}(g)\|_{L^\infty} \mathcal{A}_t(\varphi), \tag{8.11}$$

where the functional $\mathcal{A}_t(\varphi)$ is given by

$$\mathcal{A}_t(\varphi) := \int_{\mathbb{R}^n} \left(\int_0^{t/2} \fint_{B(x,s^{1/2})} |e^{(t-s)\Delta} \Delta\varphi(y)|^2 ds dy \right)^{1/2} dx.$$

The following lemma provides estimates of $\mathcal{A}_t$.

Lemma 8.8. *Let $\varphi \in \dot{H}^{1,1}$. Then*

$$\sup_{t>0} \mathcal{A}_t(\varphi) \lesssim \|\varphi\|_{\dot{H}^{1,1}}. \tag{8.12}$$

Moreover,

$$\lim_{t \to 0+} \mathcal{A}_t(\varphi) = \lim_{t \to \infty} \mathcal{A}_t(\varphi) = 0. \tag{8.13}$$

We first prepare a technical lemma. Denote by $G_t(x)$ the heat kernel

$$G_t(x) := \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}.$$

Lemma 8.9. *Let $\psi \in C^\infty(\mathbb{R}^n)$ be so that $\psi_t(x) := t^{-n/2}\psi(t^{-1/2}x)$ satisfies pointwise Gaussian estimates, i.e.,*

$$|\psi_t(x)| \lesssim \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} = G_t(x), \quad t > 0,\ x \in \mathbb{R}^n. \tag{8.14}$$

Then for any $a \in \dot{H}^{0,1}$,

$$\sup_{t>0} \int_{\mathbb{R}^n} \left(G_{3t/4} * |\psi_{t/4} * a|^2\right)^{1/2}(x)dx \lesssim \|a\|_{\dot{H}^{0,1}}. \tag{8.15}$$

Proof. Recall that $\dot{H}^{0,1}$ identifies with the Hardy space $H^1(\mathbb{R}^n)$. We say a function a on $\mathbb{R}^n$ is an $\dot{H}^{0,1}$-*atom* if there exists a ball $B \subset \mathbb{R}^n$ so that $\mathrm{supp}(a) \subset B$, $\|a\|_{L^\infty} \le |B|^{-1}$, and $\int_{\mathbb{R}^n} a(x)dx = 0$. By atomic decomposition of Hardy spaces (see e.g. [Ste93, §III.2.2, Theorem 2]), it suffices to prove that there is a uniform constant $C > 0$ so that for any a as an $\dot{H}^{0,1}$-atom,

$$\sup_{t>0} \int_{\mathbb{R}^n} \left(G_{3t/4} * |\psi_{t/4} * a|^2\right)^{1/2}(x)dx \le C. \tag{8.16}$$

Let a be an $\dot{H}^{0,1}$-atom and B be the ball so that $\mathrm{supp}(a) \subset B$ and $\|a\|_{L^\infty} \le |B|^{-1}$. By translation, we may assume that B is centered at 0. Note that for any ball B centered at 0, direct computation shows

$$\frac{\mathbb{1}_B}{|B|}(x) \lesssim G_{r(B)^2}(x), \quad x \in \mathbb{R}^n. \tag{8.17}$$

We hence get $|a(x)| \le \frac{\mathbb{1}_B}{|B|}(x) \lesssim G_{r(B)^2}(x)$ for any $x \in \mathbb{R}^n$. Then using (8.14) and the semigroup property, we obtain that for any $t > 0$,

$$\begin{aligned}\int_{\mathbb{R}^n} \left(G_{3t/4} * |\psi_{t/4} * a|^2\right)^{1/2}(x)dx &\lesssim \int_{\mathbb{R}^n} \left(G_{3t/4} * |G_{t/4} * G_{r(B)^2}|^2\right)^{1/2}(x)dx \\ &\lesssim \int_{\mathbb{R}^n} \left(G_{3t/4} * G^2_{t/4+r(B)^2}\right)^{1/2}(x)dx \\ &\lesssim \left(\frac{7t + 4r(B)^2}{2t + 8r(B)^2}\right)^{n/4},\end{aligned}$$

which is uniformly bounded by a constant independent of t and $r(B)$. This proves (8.16) and hence completes the proof. □

Now, we provide the proof of Lemma 8.8.

Proof of Lemma 8.8. Let $t > 0$, $x \in \mathbb{R}^n$, and $0 < s < t/2$. Jensen's inequality yields

$$|e^{(t-s)\Delta}\Delta\varphi|^2(x) \le e^{(\frac{3}{4}t-s)\Delta}|e^{\frac{1}{4}t\Delta}\Delta\varphi|^2(x).$$

Using (8.17) and the semigroup property of the heat kernel, we get

$$\begin{aligned}\fint_{B(x,s^{1/2})} |e^{(t-s)\Delta}\Delta\varphi(y)|^2 dy &\leq \int_{\mathbb{R}^n} \frac{\mathbb{1}_{B(0,s^{1/2})}}{|B(0,s^{1/2})|}(x-y)e^{(\frac{3}{4}t-s)\Delta}|e^{\frac{1}{4}t\Delta}\Delta\varphi|^2(y)dy \\ &\lesssim \int_{\mathbb{R}^n} G_s(x-y)e^{(\frac{3}{4}t-s)\Delta}|e^{\frac{1}{4}t\Delta}\Delta\varphi|^2(y)dy \\ &\lesssim e^{\frac{3}{4}t\Delta}|e^{\frac{1}{4}t\Delta}\Delta\varphi|^2(x).\end{aligned}$$

Applying this estimate on $\mathcal{A}_t(\varphi)$ gives

$$\mathcal{A}_t(\varphi) \lesssim \int_{\mathbb{R}^n} \left(te^{\frac{3}{4}t\Delta}|e^{\frac{1}{4}t\Delta}\Delta\varphi|^2\right)^{1/2}(x)dx. \tag{8.18}$$

To prove (8.12), since the kernel of $(t^{1/2}e^{t\Delta}\operatorname{div})_{t>0}$ satisfies pointwise Gaussian estimates (cf. (8.14)), we infer from Lemma 8.9 that

$$\mathcal{A}_t(\varphi) \lesssim \int_{\mathbb{R}^n} \left(e^{\frac{3}{4}t\Delta}|t^{1/2}e^{\frac{1}{4}t\Delta}\operatorname{div}\nabla\varphi|^2\right)^{1/2}(x)dx \lesssim \|\nabla\varphi\|_{\dot{H}^{0,1}} \eqsim \|\varphi\|_{\dot{H}^{1,1}}.$$

The implicit constant is independent of t, so (8.12) follows by taking supremum over $t>0$.

To prove (8.13), by density of $\mathscr{S}_\infty$ in $\dot{H}^{1,1}$, it suffices to prove the limit for $\varphi \in \mathscr{S}_\infty$, where in particular, we have $\varphi \in \dot{H}^{0,1}$ and $\Delta\varphi \in \dot{H}^{0,1}$. Then (8.18) gives

$$\mathcal{A}_t(\varphi) \lesssim t^{1/2}\int_{\mathbb{R}^n} \left(G_{3t/4} * |G_{t/4} * \Delta\varphi|^2\right)^{1/2}(x)dx \lesssim t^{1/2}\|\Delta\varphi\|_{\dot{H}^{0,1}},$$

which tends to 0 as $t \to 0+$. On the other hand, using the fact that the kernel of $(t\Delta e^{t\Delta})_{t>0}$ satisfies pointwise Gaussian estimates, we get

$$\mathcal{A}_t(\varphi) \lesssim t^{-1/2}\int_{\mathbb{R}^n} \left(e^{\frac{3}{4}t\Delta}|t\Delta e^{\frac{1}{4}t\Delta}\varphi|^2\right)^{1/2}(x)dx \lesssim t^{-1/2}\|\varphi\|_{\dot{H}^{0,1}},$$

which tends to 0 as $t \to \infty$. This proves (8.13). □

8.4.1.2 Estimates of R_1

We first prepare a lemma.

Lemma 8.10. *Let $0<\sigma<\tau<\infty$. For any $g \in L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)$, the function $\widetilde{g}(s,y) := g(\sigma+s,y)$ satisfies*

$$\|\mathcal{C}^{(2)}_{\tau-\sigma}(\widetilde{g})\|_{L^\infty} \lesssim (\tau-\sigma)^{1/2}\sigma^{-1/2}\|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}.$$

Proof. This lemma is due to [ADT04, Lemma 7]. For sake of completeness, we provide the computation. Let B be a ball in $\mathbb{R}^n$ with $r(B)^2 < \tau - \sigma$. Then

$$\begin{aligned}\int_0^{r(B)^2} \fint_B |\widetilde{g}(s,y)|^2 dsdy &= \int_0^{r(B)^2} \fint_B (\sigma+s)^{-1} |(\sigma+s)^{\frac{1}{2}} g(\sigma+s,y)|^2 dsdy \\ &\le \sigma^{-1}(\tau-\sigma) \|g\|^2_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}.\end{aligned}$$

Taking supremum on such B gives the desired formula. □

We now provide estimates of R_1. Define $\widetilde{g}(s,y) := g(s+t/2,y)$. Using change of variable, we have

$$R_1 = \int_0^{t/2} \int_{\mathbb{R}^n} |\widetilde{g}(s,y)| |e^{(\frac{t}{2}-s)\Delta} \Delta\varphi(y)| dsdy.$$

Similarly, we also split R_1 into two parts, the main term

$$\Phi_2 := \int_0^{t/4} \int_{\mathbb{R}^n} |\widetilde{g}(s,y)| |e^{(\frac{t}{2}-s)\Delta} \Delta\varphi(y)| dsdy,$$

and the remainder for second iteration

$$R_2 := \int_{t/4}^{t/2} \int_{\mathbb{R}^n} |\widetilde{g}(s,y)| |e^{(\frac{t}{2}-s)\Delta} \Delta\varphi(y)| dsdy.$$

Then the same deduction as Φ_1 (cf. (8.11)) implies

$$\Phi_2 \lesssim \|\mathcal{C}^{(2)}_{t/4}(\widetilde{g})\|_{L^\infty} \mathcal{A}_{t/2}(\varphi) \lesssim \left(\frac{2^{-2}}{1-2^{-1}}\right)^{1/2} \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)} \mathcal{A}_{t/2}(\varphi).$$

The last inequality comes from applying Lemma 8.10 on $\sigma = (1-2^{-1})t$ and $\tau = (1-2^{-2})t$. Therefore, by iteration, we obtain

$$\begin{aligned}R_1 &= \sum_{k=2}^\infty \int_0^{2^{-k}t} \int_{\mathbb{R}^n} \left| g\left((1-2^{-(k-1)})t+s,y\right)\right| |e^{(2^{-(k-1)}t-s)\Delta} \Delta\varphi(y)| dsdy \\ &\lesssim \sum_{k=2}^\infty 2^{-k/2} \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)} \mathcal{A}_{2^{-k+1}t}(\varphi).\end{aligned}$$

Combining it with (8.11) and (8.12) gives

$$\begin{aligned}\Phi &\lesssim \|\mathcal{C}^{(2)}_{t/2}(g)\|_{L^\infty} \mathcal{A}_t(\varphi) + \sum_{k=2}^\infty 2^{-k/2} \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)} \mathcal{A}_{2^{-k+1}t}(\varphi) \\ &\lesssim (\|g\|_{T^{\infty,2}} + \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}) \|\varphi\|_{\dot{H}^{1,1}}.\end{aligned} \tag{8.19}$$

Moreover, for each $k \ge 1$, Lemma 8.8 ensures that $\mathcal{A}_{2^{-k+1}t}(\varphi)$ tends to 0 as $t \to 0$. Since the series

$$\sum_{k=2}^\infty 2^{-k/2} \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)} \|\varphi\|_{\dot{H}^{1,1}}$$

converges, Lebesgue's dominated convergence theorem yields Φ tends to 0 as $t \to 0+$. This proves (8.10).

Meanwhile, by density of $\mathscr{S}_\infty$ in $\dot{H}^{1,1}$, we get from (8.19) that for any $t > 0$ and $g \in L^\infty_{-1/2}(\mathbb{R}^{1+n}_+) \cap T^{\infty,2}$, $\mathcal{L}_0(g)(t)$ lies in BMO^{-1} with

$$\sup_{t>0} \|\mathcal{L}_0(g)(t)\|_{\mathrm{BMO}^{-1}} \lesssim \|g\|_{T^{\infty,2}} + \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}. \tag{8.20}$$

Therefore, applying (8.10) and a density argument gives that $\mathcal{L}_0(g)(t)$ tends to 0 in BMO^{-1} as $t \to 0+$.

8.4.2 Continuity at $t > 0$

Next, we consider the continuity at $t > 0$. Without loss of generality, we show that $\mathcal{L}_0(g)(t+h) - \mathcal{L}_0(g)(t)$ tends to 0 in BMO^{-1} as $h \to 0+$. The other side follows similarly. Fix $\varphi \in \mathscr{S}_\infty$. Fubini's theorem again yields

$$\begin{aligned}
&|\langle \mathcal{L}_0(g)(t+h) - \mathcal{L}_0(g)(t), \varphi \rangle_{L^2(\mathbb{R}^n)}| \\
&\quad \leq \int_0^t \int_{\mathbb{R}^n} |g(s,y)| \left| e^{(t-s)\Delta}(e^{h\Delta} - \mathbb{I})\Delta\varphi(y) \right| dsdy \\
&\qquad + \int_t^{t+h} \int_{\mathbb{R}^n} |g(s,y)| \left| e^{(t+h-s)\Delta}\Delta\varphi(y) \right| dsdy =: I + II.
\end{aligned}$$

For I, using (8.19), we have

$$I \lesssim (\|g\|_{T^{\infty,2}} + \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}) \|(e^{h\Delta} - \mathbb{I})\varphi\|_{\dot{H}^{1,1}}.$$

Since the heat semigroup is strongly continuous on $\dot{H}^{1,1}$, we know that

$$\|(e^{h\Delta} - \mathbb{I})\varphi\|_{\dot{H}^{1,1}}$$

tends to 0 as $h \to 0+$, and so does I.

For II, using change of variable, one gets

$$II = \int_0^h \int_{\mathbb{R}^n} |g(t+s,y)| |e^{(h-s)\Delta}\Delta\varphi(y)| dsdy.$$

Write $\widetilde{g}(s,y) := g(t+s,y)$. Lemma 8.10 implies that

$$\|\mathcal{C}^{(2)}_{h/2}(\widetilde{g})\|_{L^\infty} \lesssim h^{1/2} t^{-1/2} \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)},$$

and direct computation shows

$$\|\widetilde{g}\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)} \lesssim \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}.$$

So applying (8.19) to $t = h$ and $g = \widetilde{g}$, we obtain

$$\begin{aligned} II &\lesssim \|\mathcal{C}^{(2)}_{h/2}(\widetilde{g})\|_{L^\infty}\mathcal{A}_h(\varphi) + \sum_{k=2}^{\infty} 2^{-k/2}\|\widetilde{g}\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}\mathcal{A}_{2^{-k+1}h}(\varphi) \\ &\lesssim \|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}\left(h^{1/2}t^{-1/2}\mathcal{A}_h(\varphi) + \sum_{k=2}^{\infty} 2^{-k/2}\mathcal{A}_{2^{-k+1}h}(\varphi)\right), \end{aligned}$$

which also converges to 0 as $h \to 0+$ by Lebesgue's dominated convergence theorem. A density argument thus yields $\mathcal{L}_0(g)(t+h)$ converges to $\mathcal{L}_0(g)(t)$ in BMO^{-1} as $h \to 0+$.

8.4.3 Long-time limit

We show that $\mathcal{L}_0(g)(t)$ tends to 0 in BMO^{-1} as $t \to \infty$. Pick $\varphi \in \mathscr{S}_\infty$. We infer from (8.19) that

$$|\langle \mathcal{L}_0(g)(t), \varphi\rangle| \lesssim \|g\|_{T^{\infty,2}}\mathcal{A}_t(\varphi) + \sum_{k=2}^{\infty} 2^{-k/2}\|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}\mathcal{A}_{2^{-k+1}t}(\varphi).$$

Using Lemma 8.8 again, we get for any $k \geq 1$, $\mathcal{A}_{2^{-k+1}t}(\varphi) \lesssim \|\varphi\|_{\dot{H}^{1,1}}$ and $\mathcal{A}_{2^{-k+1}t}(\varphi)$ tends to 0 as $t \to \infty$. Ensured by the convergence of the series

$$\sum_{k=2}^{\infty} 2^{-k/2}\|g\|_{L^\infty_{-1/2}(\mathbb{R}^{1+n}_+)}\|\varphi\|_{\dot{H}^{1,1}},$$

Lebesgue's dominated convergence theorem yields that $\langle \mathcal{L}_0(g)(t), \varphi\rangle$ tends to 0 as $t \to \infty$. By density, we get $\mathcal{L}_0(g)$ tends to 0 in BMO^{-1} as $t \to \infty$.

This completes the proof.

8.5 Proof of Proposition 8.7

This section is devoted to proving Proposition 8.7. Again, BMO^{-1} is endowed with the weak*-topology with respect to $\dot{H}^{1,1}$.

8.5.1 Continuity at $t = 0$

Pick $\varphi \in \mathscr{S}_\infty$. Fubini's theorem yields

$$\langle \mathcal{L}_1(f)(t), \varphi\rangle_{L^2(\mathbb{R}^n)} = -\int_0^t \int_{\mathbb{R}^n} f(s,y)e^{(t+s)\Delta}\mathbb{P}\nabla\varphi(y)dsdy.$$

Write $b := \mathbb{P}\nabla\varphi$ and we get

$$\begin{aligned} |\langle \mathcal{L}_1(f)(t), \varphi\rangle_{L^2(\mathbb{R}^n)}| \leq &\int_0^t \int_{\mathbb{R}^n} |f(s,y)||e^{s\Delta}b(y)|dsdy \\ &+ \int_0^t \int_{\mathbb{R}^n} |f(s,y)||e^{s\Delta}(e^{t\Delta} - \mathbb{I})b(y)|dsdy. \end{aligned}$$

Denote by I and II the first term and the second term on the right-hand side, respectively.

For I, we claim that the function $(s,y) \mapsto |f(s,y)||e^{s\Delta}b(y)|$ belongs to $L^1(\mathbb{R}^{1+n}_+)$. If it holds, then Lebesgue's dominated convergence theorem yields that I tends to 0 as $t \to 0+$.

Let us verify the claim. Using duality of tent spaces cf. (8.5), we get

$$\int_0^\infty \int_{\mathbb{R}^n} |f(s,y)||e^{s\Delta}b(y)|dsdy \lesssim \|f\|_{T^{\infty,1}} \|e^{s\Delta}b\|_{T^{1,\infty}}. \tag{8.21}$$

Since $\mathbb{P}$ is bounded on the Hardy space $H^1(\mathbb{R}^n) \simeq \dot{H}^{0,1}$, we know that $b = \mathbb{P}\nabla\varphi$ lies in $\dot{H}^{0,1}$. Thanks to [FS72], we get $e^{s\Delta}b$ lies in $T^{1,\infty}$ with

$$\|e^{s\Delta}b\|_{T^{1,\infty}} \lesssim \|b\|_{\dot{H}^{0,1}} \lesssim \|\varphi\|_{\dot{H}^{1,1}}. \tag{8.22}$$

The claim hence follows. This gives the desired result for I.

For II, the above reasoning yields

$$II \lesssim \|f\|_{T^{\infty,1}} \|(e^{t\Delta} - \mathbb{I})b\|_{\dot{H}^{0,1}}.$$

Since the heat semigroup $(e^{t\Delta})$ is strongly continuous on $\dot{H}^{0,1}$, one gets

$$\|(e^{t\Delta} - \mathbb{I})b\|_{\dot{H}^{0,1}}$$

tends to 0 as $t \to 0+$, and so does II.

In fact, by density of $\mathscr{S}_\infty$ in $\dot{H}^{1,1}$, the above reasoning also implies that for any $t > 0$ and $f \in T^{\infty,1}$, $\mathcal{L}_1(f)(t)$ belongs to BMO^{-1} with

$$\sup_{t>0} \|\mathcal{L}_1(f)(t)\|_{\mathrm{BMO}^{-1}} \lesssim \|f\|_{T^{\infty,1}}. \tag{8.23}$$

A density argument shows $\mathcal{L}_1(f)(t)$ tends to 0 in BMO^{-1} as $t \to 0+$.

Remark 8.11. Combining (8.20) and (8.23), we rediscover the inequality (8.7). But the argument in [ADT04, Lemma 8] is much simpler.

8.5.2 Continuity at $t > 0$

Now fix $t > 0$. Again, it is enough to prove that $\mathcal{L}_1(f)(t+h) - \mathcal{L}_1(f)(t)$ tends to 0 in BMO^{-1} as $h \to 0+$. Pick $\varphi \in \mathscr{S}_\infty$ and write $b = \mathbb{P}\nabla\varphi$. Fubini's theorem yields

$$\begin{aligned}
&|\langle \mathcal{L}_1(f)(t+h) - \mathcal{L}_1(f)(t), \varphi \rangle_{L^2(\mathbb{R}^n)}| \\
&\quad \leq \int_0^{t+h} \int_{\mathbb{R}^n} |f(s,y)||e^{s\Delta}(e^{(t+h)\Delta} - e^{t\Delta})b(y)|dsdy \\
&\qquad + \int_t^{t+h} \int_{\mathbb{R}^n} |f(s,y)||e^{(t+s)\Delta}b(y)|dsdy =: I + II.
\end{aligned}$$

For I, we infer from (8.21) and (8.22) that

$$I \lesssim \|f\|_{T^{\infty,1}} \|(e^{(t+h)\Delta} - e^{t\Delta})b\|_{\dot{H}^{0,1}}.$$

Using strong continuity of $(e^{t\Delta})$ on $\dot{H}^{0,1}$, we get $\|(e^{(t+h)\Delta} - e^{t\Delta})b\|_{\dot{H}^{0,1}}$ tends to 0 as $h \to 0+$, and so is I.

For II, uniform boundedness of $(e^{t\Delta})$ on $\dot{H}^{0,1}$ implies that the function $(s,y) \mapsto |f(s,y)||e^{(t+s)\Delta}b(y)|$ belongs to $L^1(\mathbb{R}^{1+n}_+)$ as

$$\int_0^\infty \int_{\mathbb{R}^n} |f(s,y)||e^{(t+s)\Delta}b(y)|dsdy \lesssim \|f\|_{T^{\infty,1}} \|e^{t\Delta}b\|_{\dot{H}^{0,1}} \lesssim \|f\|_{T^{\infty,1}} \|b\|_{\dot{H}^{0,1}}.$$

Lebesgue's dominated convergence theorem hence yields II tends to 0 as $h \to 0+$. A density argument concludes that $\mathcal{L}_1(f)(t+h)$ converges to $\mathcal{L}_1(f)(t)$ in BMO^{-1} as $h \to 0+$.

8.5.3 Long-time limit

Let us finish by proving $\mathcal{L}_1(f)(t)$ tends to 0 in BMO^{-1} as $t \to \infty$. Pick $\varphi \in \mathscr{S}_\infty$ and write $b = \mathbb{P}\nabla\varphi$. Note that

$$\begin{aligned} |\langle \mathcal{L}_1(f)(t), \varphi \rangle_{L^2(\mathbb{R}^n)}| &\le \int_0^t \int_{\mathbb{R}^n} |f(s,y)||e^{(t+s)\Delta}b(y)|dsdy \\ &\lesssim \|f\|_{T^{\infty,1}} \|e^{t\Delta}b\|_{\dot{H}^{0,1}}. \end{aligned}$$

As $b \in \dot{H}^{0,1}$, the function $t \mapsto e^{t\Delta}b$ tends to 0 in $\dot{H}^{0,1}$ as $t \to \infty$, so by density again, we obtain that $\mathcal{L}_1(f)(t)$ tends to 0 in BMO^{-1} as $t \to \infty$.

This completes the proof.

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